Proability and Statistics

MA20205

Bibhas Adhikari

Autumn 2022-23, IIT Kharagpur

Lecture 2

August 16, 2022

Probability

- Probability theory - Kolmogorov
- Probability theory $\mapsto$ Mathematical statistics

Probability

- ☐ Probability theory - Kolmogorov
- ☐ Probability theory $\mapsto$ Mathematical statistics

Sample space

- Discrete sample space - countable sample points
- Continuous sample space - uncountable sample points

- ☐ Example: roll a pair of dice:

Probability

- ☐ Probability theory - Kolmogorov
- ☐ Probability theory $\mapsto$ Mathematical statistics

Sample space

- Discrete sample space - countable sample points
- Continuous sample space - uncountable sample points

- ☐ Example: roll a pair of dice:

$$S = \{(x, y) : 1 \leq x \leq 6, 1 \leq y \leq 6\}$$

- Events: sum of the outcomes is equal to 8

Probability

σ-field

A set $A \subseteq S$ is an event if $A \in \mathcal{F} \subset \mathcal{P}(S)$, power set of S such that

- $S \in \mathcal{F}$
- if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$
- if $A_j \in \mathcal{F}$, $j \geq 1$ then $\bigcup_{j=1}^{\infty} A_j \in \mathcal{F}$

Question: Why do we need the notion of σ-field?

Probability

Question: Probability of what?

Probability

Question: Probability of what?

- Frequentist Interpretation of Probability and classical definition

$$P(A) = \frac{n(A)}{n(S)} = \frac{\#A}{\#S}$$

Probability

Question: Probability of what?

- Frequentist Interpretation of Probability and classical definition

$$P(A) = \frac{n(A)}{n(S)} = \frac{\#A}{\#S}$$

Probability measure (axiomatic definition)

Probability is function $P : \mathcal{F} \to [0\ 1]$ - it assigns nonnegative numbers to events such that

- $P(S) = 1$
- (addition rule)

$$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$$

if $A_1, A_2, \ldots, A_k, \ldots$ are disjoint events of S

Probability

Properties of probability

- $P(\emptyset) = 0$
- For any finite collection of mutually exclusive events: $\{A_1, A_1, \ldots, A_n\}$ with $A_i \cap A_j = \emptyset$ if $i \neq j$ then

$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A_k)$$

- For a discrete sample space $S = \{s_1, s_2, \ldots\}$,

$$P(S) = \sum_{k=1}^{\infty} P(\{s_k\})$$

Probability

Properties of probability

- $P(\emptyset) = 0$
- For any finite collection of mutually exclusive events: $\{A_1, A_1, \ldots, A_n\}$ with $A_i \cap A_j = \emptyset$ if $i \neq j$ then

$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A_k)$$

- For a discrete sample space $S = \{s_1, s_2, \ldots\}$,

$$P(S) = \sum_{k=1}^{\infty} P(\{s_k\})$$

Question: Does $P(A) = 1$ imply $A = S$?

Probability

Question: Is the axiomatic definition a more general definition than the classical one?

Probability

Question: Is the axiomatic definition a more general definition than the classical one?

Derive: for equally likely sample points,

$$P(A) = \frac{\#A}{\#S}$$

Probability

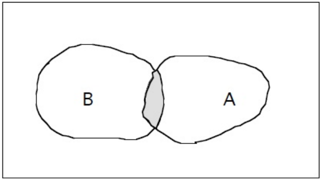
Some more properties of probability measure (use Venn diagram)

- $P(A^c) = 1 - P(A)$
- if $A \subseteq B \subseteq S$ then $P(A) \leq P(B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- Let $A_1 \subseteq A_2$. Then

$$P(A_2 \setminus A_1) = P(A_2) - P(A_1)$$

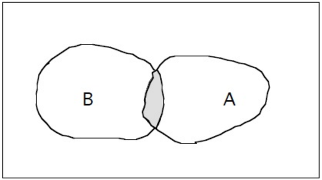
Problems: .....

Probability



Question: For a finite sample space, how do we define probability of A given that B has happened?

Probability



Question: For a finite sample space, how do we define probability of A given that B has happened?

$$P(A\,\text{given}\,B) = \frac{\text{number of elements of}\quad A \cap B}{\text{number of elements of}\, B}$$

Probability

Conditional probability

For both discrete and continuous sample space:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

provided $P(B) > 0$.

Probability

Conditional probability

For both discrete and continuous sample space:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

provided $P(B) > 0$.

Conditional probability measure:

- $P(A|B) \geq 0$ for all event A
- $P(B|B) = 1$
- (addition rule)

$$P\left(\bigcup_{k=1}^{\infty} A_k | B\right) = \sum_{k=1}^{\infty} P(A_k|B)$$

for mutually exclusive events $A_1, A_2, \ldots, A_k, \ldots$

Problems...