Proability and Statistics

MA20205

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Hypothesis Testing

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- The coin is unbiased
- Students whose registration are accepted for the *Quantum Computing* course have CGPA ≥ 8
- An algorithm performs better than another algorithm

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- H_1 : Alternative hypothesis (which is alternative to the null hypothesis)

Hypothesis Testing

Analogue of Hypothesis Testing in courthouse Suppose a person being prosecuted is assumed innocent. This is the null hypothesis. Then the police need to produce sufficient evidence to prove the person guilty. Thus the Hypothesis Testing investigates whether there are enough evidence to reject the null hypothesis.

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Two important approaches for Hypothesis Testing:

- critical-value test
- p-value test

Hypothesis Testing

Critical-value test Consider a toy example. Suppose we have a four-sided die and we want to test whether the die is unbiased. Define

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Suppose

- the die is drawn $n = 1000$ times
- 3 appears 290 times

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Let $X_1, X_2, \ldots, X_n$ denote the n copies of the Bernoulli random variable with success means 3 occurs and false means 3 does not occur.

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Suppose

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Let $X_1, X_2, \ldots, X_n$ denote the n copies of the Bernoulli random variable with success means 3 occurs and false means 3 does not occur. If the true probability is $\theta = 0.25$ then we should have

$$P(X_i = 3) = \theta = 0.25 \text{ and } P(X_i \neq 3) = 1 - \theta = 0.75$$

Hypothesis Testing

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Then the question for Hypothesis testing is: How far is $\widehat{\theta} = 0.29$ from $\theta = 0.25$? Note that as per the data H_0 should be rejected!

Hypothesis Testing

However, we should do a theoretical analysis over the sample size before rejecting H_0. If n is large, from central limit theorem,

$$\widehat{\theta} \sim \mathcal{N}\left(\theta, \frac{\sigma^2}{n}\right)$$

Recall that $\widehat{\theta}$ is an unbiased estimator!!

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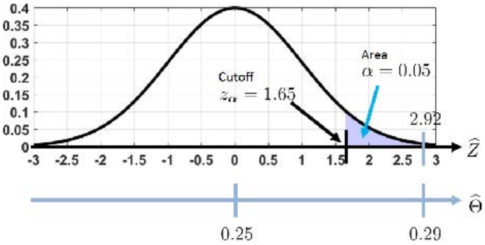
Therefore, $\widehat{\theta} = 0.29$ is equivalent to $\widehat{Z} = 2.92$, and $\widehat{\theta} = 0.25$ is equivalent to $\widehat{Z} = 0$.

Hypothesis Testing

The mapping between $\widehat{\theta}$ and $\widehat{Z}$ is plotted below:



Critical level α is chosen a small value, exp. $\alpha = 0.05$ such that the corresponding cutoff is given by

$$z_\alpha = \text{cutoff location where the area under the curve is } \alpha$$

Thus

$$\Phi(z_\alpha) = 1 - \alpha.$$

Hypothesis testing

For the example above, the cutoff $z_\alpha = z_{0.05} = 1.65$

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Critical-value test

- Set a critical value z_α, and compute $\widehat{Z} = \frac{\widehat{\theta} - \theta}{\sigma/\sqrt{n}}$
- If $\widehat{Z} \geq z_\alpha$ then reject H_0
- If $\widehat{Z} < z_\alpha$ then keep H_0

Hypothesis Testing

p-value test

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Suppose the given data is as follows: $n = 150$ tosses, and number of heads in those tosses is 128. Then

$$\widehat{\theta} = \frac{128}{150} = 0.853.$$

Hypothesis Testing

Then we have

$$\widehat{Z} = \frac{\widehat{\theta} - \theta}{\sigma/\sqrt{n}} = \frac{0.853 - 0.9}{\sqrt{\frac{0.9(1-0.9)}{150}}} = -1.92$$

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Now we find the the probability under the curve if we integrate the pdf of $\widehat{Z}$ from $-\infty$ to -1.92. Since $\widehat{Z} \sim \mathcal{N}(0,1)$, we have

$$P\left(\widehat{Z} \leq -1.92\right) = 0.0274 \text{ (}p\text{-value)}$$

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p-value

$$p - \text{value} = P(\widehat{Z} \leq z)$$

where z is the random realization of $\widehat{Z}$ estimated from the data

Hypothesis Testing

Decision rule from the p-value:

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Relation between critical-value and p-value

There is a one-one correspondence. In the p-value test, if $\widehat{Z}$ is normal then

$$p\text{-value} = P(\widehat{Z} \leq z) = \Phi(z)$$

where Φ is the CDF of $\mathcal{N}(0, 1)$. Then taking the inverse

$$z = \Phi^{-1}(p\text{-value})$$

Hypothesis Testing

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Thus if the test statistic fails for the p-value, it will also fail in the critical-value test, and vice-versa.

Hypothesis Testing

Difference between critical-value test and p-value test

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- critical-value test: compare with respect to critical value, which is the cutoff on the *z*-axis
- *p*-value test: compare with respect to α, which is the probability
- both give the same conclusion

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Suppose $X_1, \ldots, X_n$ is a random sample from a population with pdf

$$f(x;\theta) = \begin{cases} (1+\theta)x^{\theta}, \ 0 < x < 1 \\ 0, \ \text{otherwise}, \end{cases}$$

where $\theta > 0$ is an unknown parameter.

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Let $n = 4$ and $x_1 = 0.92$, $x_2 = 0.75$, $x_3 = 0.85$, $x_4 = 0.8$ is a sample data from the above distribution. Applying ML method, the estimator $\widehat{\theta}$ of θ is

$$\widehat{\theta} = -1 - \frac{4}{\ln(X_1) + \ln(X_2) + \ln(X_3) + \ln(X_4)}$$

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hence the ML estimate of θ is

$$\widehat{\theta} = -1 - \frac{4}{\ln(0.92) + \ln(0.75) + \ln(0.85) + \ln(0.80)} = 4.2861.$$

Hypothesis Testing

We we denote the null hypothesis and alternative hypothesis as

$$H_o : \theta \in \Omega_0 \text{ and } H_a : \theta \in \Omega_a$$

where Ω_o and Ω_a are subsets of the parameter space Ω with

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The *likelihood ratio test statistic* for testing the null hypothesis $H_o : \theta \in \Omega_o$ against the alternative hypothesis $H_a : \theta \notin \Omega_o$ based on a set of random sample data $x_1, x_2, \ldots, x_n$ is defined as

$$W(x_1, \ldots, x_n) = \frac{\max_{\theta \in \Omega_o} L(\theta, x_1, x_2, \ldots, x_n)}{\max_{\theta \in \Omega} L(\theta, x_1, x_2, \ldots, x_n)}$$

where

$$L(\theta, x_1, \ldots, x_n) = \prod_{i=1}^{n} f(x_i, \theta)$$

Hypothesis Testing

The critical region C (a Borel set in $\mathbb{R}^n$) of a test statistic $W(X_1, X_2, \ldots, X_n)$ is such that if $W(x_1, \ldots, x_n)$ is an element of C then we decide to accept H_a; otherwise we accept H_o.

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	H_o is true	H_o is false
Accept H_0	Correct decision	Type II Error
Reject H_o	Type I Error	Correct Decision

Hypothesis testing

Significance level of the hypothesis test Let $H_o : \theta \in \Omega_o$ and $H_a : \theta \notin \Omega_o$ be the null hypothesis to be tested based on a random sample $X_1, \ldots, X_n$ from a population X with density $f(x;\theta)$. The *significance level* of the hypothesis test

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denoted by α, is defined as

$$\alpha = P(\text{Type I Error})$$

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Thus, the significance level of a hypothesis test we mean the probability of rejecting a true null hypothesis, that is,

$$\alpha = P(\text{Reject } H_o | H_o \text{ is true})$$

which is equivalent to

$$\alpha = P(\text{Accept } H_a | H_o \text{ is true})$$

Hypothesis Testing

The *probability of type II error* is defined as

$$\beta = P(\text{Accept } H_o | H_o \text{ is false})$$

which is equivalent to

$$\alpha = P(\text{Accept } H_o | H_a \text{ is true})$$