Proability and Statistics

MA20205

Bibhas Adhikari

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Lecture 15

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Bivariate normal distribution

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Bivariate normal distribution

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$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2}Q(x,y)}$$

where $\mu_1, \mu_2 \in \mathbb{R}$, $\sigma_1, \sigma_2 \in (0, \infty)$ and $\rho \in (-1, 1)$ are parameters, and

$$Q(x,y) = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]$$

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Denote:

$$(X, Y) \sim \mathcal{N}(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$$

Bivariate normal/Gaussian distribution

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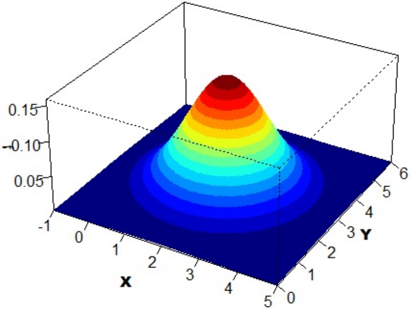
3. σ_1^2 and σ_2^2 measure the spread of the mountain in the x-direction and y-direction respectively
4. ρ determines the shape and orientation

Watch: The link



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\begin{aligned}
f_X(x) &= \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \\
f_Y(y) &= \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}
\end{aligned}
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Remark However the converse need not be true!!

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proof of marginal distribution

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Bivariate normal/Gaussian distribution

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&\quad \left. + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right] \\
&= \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_1}{\sigma_1} - \rho\frac{y-\mu_2}{\sigma_2}\right)^2 + (1-\rho^2)\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right] \\
&= \frac{(x-a)^2}{(1-\rho^2)\sigma_1^2} + \frac{(y-\mu_2)^2}{\sigma_2^2}
\end{aligned}
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where $a = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)$.

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&= \frac{(x-a)^2}{(1-\rho^2)\sigma_1^2} + \frac{(y-\mu_2)^2}{\sigma_2^2}
\end{aligned}
$$

where $a = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)$.Hence

$$
f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \frac{1}{\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}} \int_{-\infty}^{\infty} e^{-\frac{(x-a)^2}{2(1-\rho^2)\sigma_1^2}} dx
$$

Bivariate normal/Gaussian distribution

If $(X, Y) \sim \mathcal{N}(\mu_X, \mu_Y, \sigma_X, \sigma_Y, \rho)$ then

$$E(X) = \mu_X,\ E(Y) = \mu_Y,\ \mathrm{Var}(X) = \sigma_X^2,\ \mathrm{Var}(Y) = \sigma_Y^2$$

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$$\text{Correlation coefficient } = \rho,\ M(s,t) = e^{\mu_X s + \mu_Y t + \frac{1}{2}(\sigma_X^2 s^2 + 2\rho\sigma_X\sigma_Y st + \sigma_Y^2 t^2)}$$

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Proof $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$, $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$. Then
$W = sX + tY \sim \mathcal{N}(\mu_W, \sigma_W^2)$ where

$$\mu_W = s\mu_X + t\mu_Y,\ \sigma_W^2 = s^2\sigma_X^2 + 2st\rho\sigma_X\sigma_Y + t^2\sigma_Y.$$

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Therefore, the mgf of W is $M(\tau) = e^{\mu_W \tau + \frac{1}{2}\tau^2\sigma_W^2}$.

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Therefore, the mgf of W is $M(\tau) = e^{\mu_W\tau + \frac{1}{2}\tau^2\sigma_W^2}$.Then mgf of (X, Y) is

$$\begin{aligned} M(s,t) &= E(e^{sX+tY}) = e^{\mu_W + \frac{1}{2}\sigma_W^2} \\ &= e^{\mu_X s + \mu_Y t + \frac{1}{2}(\sigma_X^2 s^2 + 2\rho\sigma_X\sigma_Y st + \sigma_Y^2 t^2)} \end{aligned}$$

Bivariate normal/Gaussian distribution

Let $f(x, y)$ be the joint pdf of (X, Y). Then conditional density of Y given $X = x$ is

$$f_{Y|X}(y|x) = \frac{1}{\sigma_Y \sqrt{2\pi(1-\rho^2)}} e^{-\frac{1}{2}\left(\frac{y-b}{\sigma_Y\sqrt{1-\rho^2}}\right)^2}$$

where

$$b = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X).$$

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Similarly,

$$f_{X|Y}(x|y) = \frac{1}{\sigma_X \sqrt{2\pi(1-\rho^2)}} e^{-\frac{1}{2}\left(\frac{x-c}{\sigma_X\sqrt{1-\rho^2}}\right)^2}$$

where

$$c = \mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y - \mu_Y)$$

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$$\text{Var}(Y|x) = \sigma_Y^2(1 - \rho^2)$$
$$\text{Var}(X|y) = \sigma_X^2(1 - \rho^2)$$

Interesting result (Cramer, 1941)

Two random variables X and Y have a joint bivariate normal distribution if and only if every linear combination of X and Y has univariate normal distribution