Proability and Statistics

MA20205

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Lecture 10
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Functions of a Random Variable

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Problems...

Functions of a Random Variable

For continuous random variables Let X be a continuous random variable with pdf $f(x)$. Let $y = T(x)$ be an increasing (or decreasing) function. Then the pdf of the random variable $Y = T(X)$ is given by

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Suppose $y = T(x)$ is an increasing function.

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Then

$$g(y) = \frac{dG(y)}{dy} = \frac{d}{dy}\left(\int_{-\infty}^{W(y)} f(x)dx \right) = f(W(y))\frac{dW(y)}{dy} = f(W(y))\frac{dx}{dy}$$

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Three important functions for accessing reliability:

1. failure density, denoted by f (pdf of X)
2. the reliability function R, the probability that the system (or a component) will not fail before time t
3. the failure or hazard rate of the distribution

Reliability

Let a system being put into operation at time $t = 0$. We observe the system until it eventually fails. Then

$$\begin{aligned} R(t) &= 1 - P(\text{the system will fail beforw time } t) \\ &= 1 - \int_0^t f(x)dx \\ &= 1 - F(t) \end{aligned}$$

where F is the CDF of X.

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Hazard/failure rate ($\rho(t)$): Consider the time interval $[t,\ t+\triangle t]$ of length $\triangle$. Then

$$\begin{aligned}\rho(t) &= \frac{P(t \leq X \leq t+\triangle t | t \leq X)}{\triangle t} \\ &= \lim_{\triangle t \to 0} \frac{\int_t^{t+\triangle t} f(x)dx}{\int_t^{\infty} f(x)dx} \frac{1}{\triangle t} \\ &= \lim_{\triangle \to 0} \frac{F(t+\triangle t) - F(t)}{1-F(t)} \frac{1}{\triangle t} \\ &= \lim_{\triangle \to 0} \frac{F'(t)}{R(t)} = \frac{f(t)}{R(t)}\end{aligned}$$

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&= \lim_{\triangle \to 0} \frac{F'(t)}{R(t)} = \frac{f(t)}{R(t)}
\end{aligned}$$

Question What is the job of a reliability scientist/engineer?

Reliability

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Theorem $R(t) = \exp\left\{-\int_0^t \rho(x)dx\right\}$ and $f(t) = \rho(t)R(t)$.

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$R(x) = 1 - F(x) \Rightarrow R'(x) = -F'(x)$. Thus

$$\rho(x) = \frac{f(x)}{R(x)} = \frac{F'(x)}{R(x)} = \frac{R'(x)}{R(x)}.$$

Then

$$\int_0^t \rho(x)dx = -\int_0^t \frac{R'(x)}{R(x)} = -[\ln R(t) - \ln R(0)].$$

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Now $R(0) = 1$ since the system will not fail before t. Thus

$$\ln R(t) = -\int_0^t \rho(x)dx.$$

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Suppose a series system with k components. Let $R_i(t)$ denote the reliability of component i and assume that the components are independent i.e. one is unaffected by the reliability of others.

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Suppose a series system with k components. Let $R_i(t)$ denote the reliability of component i and assume that the components are independent i.e. one is unaffected by the reliability of others.
Then the reliability of the entire system is the probability that the system will not fail before time t. Thus the system will not fail if and only if no component fails before time t. Therefore, reliability of the system, $R_s(t)$ is given by

$$R_s(t) = \prod_{i=1}^{k} R_i(t).$$

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$$R_s(t) = 1 - P(\text{all components fail}) = 1 - \prod_{i=1}^{k}(1 - R_i(t)).$$