L-9

Probability & Statistics

~~Geometric~~ Exponential Distribution.

The r.v. W is described as
↓
waiting time in Poisson process before occurrance of an event

$$F(w) = P(W \le w)$$
$$= 1 - P(W \geq w)$$
$$= 1 - P(\text{no occurrance during } [0, w])$$
$$= 1 - e^{-\lambda w}$$

$$F'(w) = f(w) = \lambda e^{-\lambda w}.$$

~~Let~~ Setting $\theta = \frac{1}{\lambda}$

pdf of Exponential distribution ← $f(x) = \frac{1}{\theta} e^{-x/\theta}, \quad 0 \le x < \infty$ and $\theta > 0$

MGF of exponential distribution.

$$M(t) = \int_0^\infty e^{tx} \frac{1}{\theta} e^{-x/\theta} dx$$

$$= \frac{1}{1-\theta t}, \quad t < \frac{1}{\theta}$$

$$M'(t) = \frac{\theta}{(1-\theta t)^2}, \quad M''(t) = \frac{2\theta^2}{(1-\theta t)^3}$$

$$\mu = M'(0) = \theta \text{ and}$$

$$\sigma^2 = M''(0) - [M'(0)]^2 = \theta^2$$

so if λ is the mean number of occurances in the unit interval. Then $\theta = 1/\lambda$ is the mean waiting time for the first occurance.

X has the cdf.,

$$F(x) = \begin{cases} 0, & -\infty < x < 0 \\ 1 - e^{-x/\theta}, & 0 \le x < \infty . \end{cases}$$

Now let us generalize the r.v. W - which measures waiting time.

Introduce a new r.v. W which denotes waiting time until α-th occurance.

Then the cdf of this new r.v. W is

$$F(w) = P(W \le w) = 1 - P(W > w)$$

$$= 1 - P(\text{fewer than } \alpha \text{ occurence in } [0, w])$$

$$= 1 - \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k e^{-\lambda w}}{k!}$$

Since the # of occurence in the interval $[0, w]$ has a Poisson distribution.

Then

$$F'(w) = \frac{\lambda(\lambda w)^{\alpha-1}}{(\alpha-1)!}e^{-\lambda w}.$$

$$\| \quad f(w)$$

Now recall Gamma function.

$$\Gamma(t) = \int_0^\infty y^{t-1}e^{-y}\,dy, \quad 0<t.$$

Then $\Gamma(t) = (t-1)\,\Gamma(t-1)$.

In fact, if 't' is a +ve integer then $\Gamma(t) = (t-1)!$

i.e. Γ is a generalized factorial!

Then the Gamma distribution is defined as

$$f(x) = \frac{1}{\Gamma(\alpha)\,\theta^{\alpha}}\,x^{\alpha-1}e^{-x/\theta}$$

$\downarrow$

two parameters α, θ.

MGF of X → which gives Gamma distribution.

$$M(t) = \frac{1}{(1-\theta t)^\alpha}, \quad t < \frac{1}{\theta}$$

Then $\mu = \alpha\theta$ and

$\sigma^2 = \alpha\theta^2$.

Now let us discuss Gamma with a fixed set of values of α & θ.

Set $\theta = 2$ and $\alpha = \frac{r}{2}$

Then the pdf of the corresponding X

$$f(x) = \frac{1}{\Gamma(\frac{r}{2})\, 2^{r/2}} x^{r/2 - 1} e^{-x/2}, \quad 0 < x < \infty$$

chi-square distribution with 'r' degrees of freedom.

Prob.

Suppose the # of customers per hour arriving at a shop follows a Poisson process with mean 30. That is, if a minute is our unit, then $\lambda = 1/2$.

What is the probability that the shopkeeper will wait more than 5 minutes before both of the first two customers arrive?