L8

Probability & Statistics

Continuous random variable.

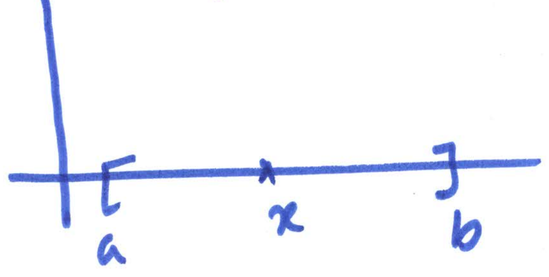
$$X : S \longrightarrow \mathbb{R}$$

range (X) → Countable (discrete)

→ not-countable (continuous)

space of $X \equiv S \equiv$ an interval in $\mathbb{R}$ or union of intervals in $\mathbb{R}$

Exp. X denotes a random variable when a point is chosen at random from an interval $[a,b]$



Assume that the point is selected from the interval $[a,x]$, $a \leq x \leq b$ with probability $\frac{x-a}{b-a}$

Then the probability is proportional to the length of the interval.

Then the cdf (cumulative distribution function) of X is

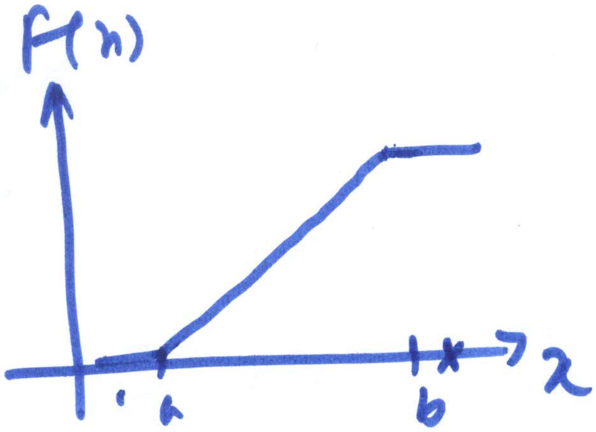
$$P(X \leq x) = F(x) = \begin{cases} 0 & , x < a \\ \frac{x-a}{b-a} & , a \leq x < b \\ 1 & , x = b \end{cases}$$

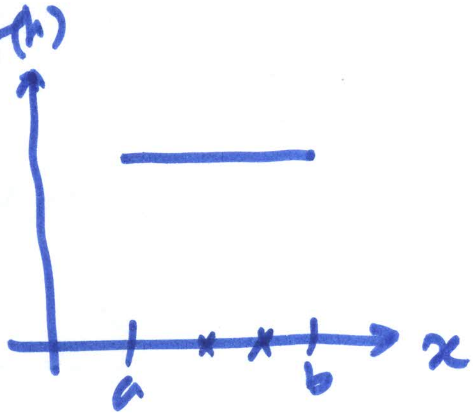
from $F(x)$ we can write it as

$$F(x) = \int_{-\infty}^{x} f(y)\, dy$$

where $f(x) = \frac{1}{b-a}$, $a \leq x \leq b$.
and $f(x) = 0$ elsewhere.

That is, $F'(x) = f(x)$ is called the probability density fn of X.





↓ is called Uniform distribution, denoted as $U(a,b)$.

Q. When does a fn. $f : \mathbb{R} \to [0,1]$ represent a pdf of some random variable?

A. Assume that f & the S is an interval or union of interval & f is integrable fn.

(a) $f(x) \geq 0$

(b) $\int_S f(x) = 1$

(c) If $(a,b) \subseteq S$
$P(a < x < b) = \int_a^b f(x)\,dx$.

Once $f(x)$ is known, then the cdf is defined as

$$F(x) = P(X \le x)$$

$$= \int_{-\infty}^{x} f(t)\,dt, \quad -\infty < x < \infty.$$

Otherwise, $F'(x) = f(x)$

<u>Exp.</u> which shows that pdf $f(x)$ need not be a 'continuous' function, however it represents a pdf.

$$f(x) = \begin{cases} \frac{1}{2}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Expectation/mean of a Continuous r.v.

$$E(X) = \mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

Variance.

$$\operatorname{var}(X) = E\left[(\cancel{x} - \mu)^2\right] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

Stainard deviation

$$SD(X) = \sqrt{\operatorname{var}(X)}$$

MGf.

$$M(t) = E(e^{tX})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$E = M'(0)$, $\operatorname{var}(x) = M''(0) - [M'(0)]^2$

$= E(X^2) - (E(X))^2$

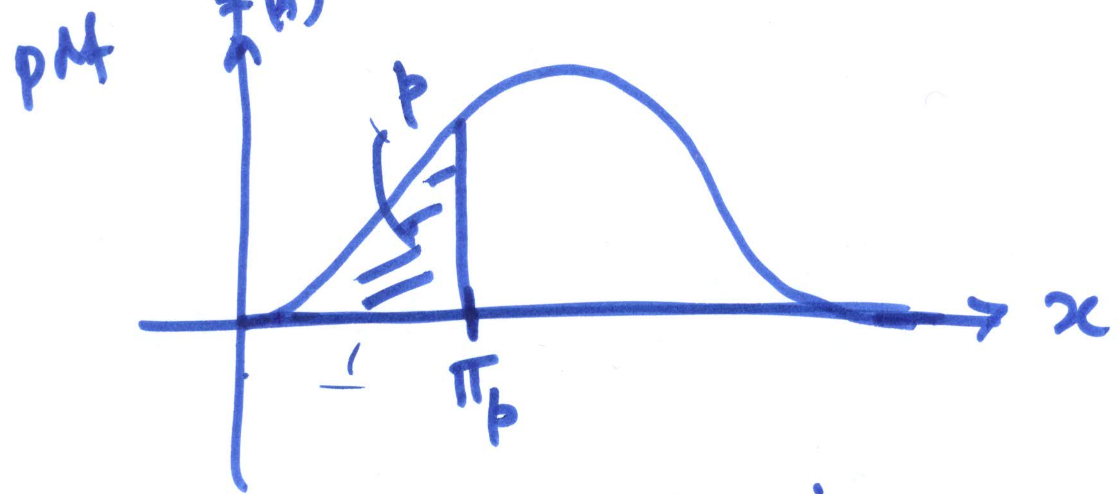
Q. Calculate Mgf, μ, var of $U(a,b)$.

Ans.

$$M(t) = \begin{cases} \dfrac{e^{tb} - e^{ta}}{t(b-a)}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

$$\mu = \frac{a+b}{2}$$

$$\sigma^2 = \frac{(b-a)^2}{12}$$



The $(100p)$-th percentile is a number π_p such that area under $f(x)$ to the left of π_p is p. That is

$$p = \int_{-\infty}^{\pi_p} f(x)\,dx = F(\pi_p)$$

If $p = 0.5$, it is called median or second quartile.

Then 25th and 75th percentiles are called first and third quartiles.

<u>Exc.</u> Let X ~~be~~ have the pdf $f(x) = e^{-x-1}$, $-1 < x < \infty$.

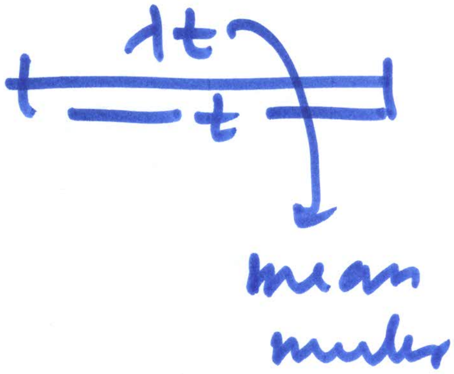
Determine:

1. $P(X \geq 1)$
2. MGF of X.
3. $\mu(X)$, $\sigma^2(X)$
4. $F(x)$
5. $\pi_{0.5}$ = median.

Recall Poisson Distribution.

λ = mean number of occurrances in a unit interval.

Let the length of the interval is 't'. Then



$$p(x) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}$$

(Verify)!! <u>H.W.</u>

Let W denote the waiting time untill the first ~~outcome~~ occurrance during the observation of a <u>Poisson</u> <u>process</u>.

Then $w \in \mathbb{R}_+$

$$F(w) = P(W \leq w)$$

$$= 1 - P(W > w)$$

$$= 1 - P(\text{no occurrence in } [0, w])$$

$$= 1 - e^{-\lambda w}$$

→ (see last page) the length is w here

Then when $w > 0$

$$F'(w) = \lambda e^{-\lambda w} = f(w)$$

Now letting $\lambda = \frac{1}{\theta}$, we say that the random variable X has an <u>exponential distribution</u> if its pdf is given by

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, \quad 0 \leq x < \infty$$