Probability & Statistics.

L-5

Let X be a discrete random variable

moment generating fn. (mgf)

$$= E(e^{tX})$$

$$= \sum_{x_i \in S_x} e^{t x_i} f(x_i)$$

$$= \sum_{x \in S} e^{tx} f(x)$$

$< \infty$, for some $-h < t < h$

$\Rightarrow$ mgf exists.

<u>Idea</u>: mgf uniquely determines the corresponding pmf.

let $S = \{ s_1, s_2, \ldots \}$

Suppose X & Y are two random variables on S, ~~and then~~ $f(x)$, $g(y)$ are the corresponding pmfs.

If mgfs are same,

$$\sum_i e^{ts_i} f(s_i) = \sum e^{ts_i} g(s_i)$$

$$\Rightarrow \quad e^{ts_1} f(s_1) + e^{ts_2} f(s_2) + \cdots$$

$$= e^{ts_1} g(s_1) + e^{ts_2} g(s_2) + \cdots \text{, for some 't'}$$

why?

$$\Rightarrow \quad \boxed{f(s_i) = g(s_i)} \quad \forall \text{'}i\text{'}.$$

Q. Let $M(t) = \dfrac{e^{t}/2}{1-e^{t}/2}$, $t < \ln 2$

be mgf of a r.v. X. Then find out the corresponding pmf.

Ans. $f(x) = \left(\frac{1}{2}\right)^x$.

(Hint) expand M(t) in terms of a series $\left(\left[1 - \frac{e^t}{2}\right]^{-1} = ? \right)$

$$\frac{e^t}{2}\left(1 + \frac{e^t}{2} + \frac{e^{2t}}{2^2} + \cdots\right)$$

$$= (e^t)\left(\tfrac{1}{2}\right) + (e^{2t})\left(\tfrac{1}{2}\right)^2 + (e^{3t})\left(\tfrac{1}{2}\right)^3 + \cdots$$

Q. what other information about x can be obtained from $M(t)$ mgf ?

$$M(t) = \sum_{x \in S} e^{tx} f(x)$$

$$\frac{d}{dt} M(t) = M'(t) = \sum_{x \in S} x e^{tx} f(x)$$

$$M''(t) = \frac{d^2}{dt^2} M(t) = \sum_{x \in S} x^2 e^{tx} f(x).$$

At $t = 0$, $M'(0) = \sum x f(x) = \mu_x = E(x)$

$$M''(0) = \sum x^2 f(x) = \mu_{x^2}$$

$$= E(x^2).$$

$$\sigma^2 = E(x^2) - (E(x))^2$$

$$= M''(0) - M'(0).$$

Q. Calculate μ & σ^2 of the Geometric distribution using its mgf.

Ans. $f(x) = q^{x-1} p$, $x \in \{1, 2, 3, \ldots\}$
$q = 1-p$.

$M(t) = E(e^{tx})$.

$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} p$

$= \dfrac{pe^t}{1-qe^t}$, provided $qe^t < 1$

i.e. $t < \underbrace{-\ln q}_{>0}$

$M'(0) = \mu = \dfrac{1}{p}$

$M''(0) = \dfrac{1+q}{p^2}$

$\sigma^2 = \dfrac{1+q}{p^2} - \dfrac{1}{p^2} = \dfrac{q}{p^2}$.

Another distribution.

Bernoulli expriment

$S = \{\text{success}, \text{failure}\}$.

Let X be the random variable associated with a Bernoulli trial.

$X(\text{success}) = 1$

$X(\text{failure}) = 0$.

Then, let $p(1) = p$

$p(0) = 1-p = q$

Then pmf of X is

$$P(X=x) = f(x) = \begin{cases} p^x (1-p)^{1-x}, & x \in \{0,1\} \\ 0, & \text{otherwise} \end{cases}$$

Q. Calculate μ & σ^2.

$\mu = p$, $\sigma^2 = pq$.