L-4

Probability & Statistics.

$S \to$ sample space

$X : S \to \mathbb{R}$

pmf $\quad f(x) = P(X = x)\ , \quad x \in \mathbb{R}$

$F(x) = P(X \leq x)$

Uniform distribution

(probability model) → comes from all outcomes which are equally likely.

Another probability model

Consider a collection $N = N_1 + N_2$ number of objects, N_1 of them belong to one of two dichotomous classes (exp. red chips) and N_2 of them belong to the second class (exp. green chips).

Let 'n' objects are selected from these N objects at random and '*without replacement*'

Find that exactly x of these n objects belong to the first class, and $n-x$ belong to the second class.

$$P(X=x) = \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}}$$

↓ 'hypergeometric distribution'

$S = \{x_1, \ldots, x_n\}$, $p_i = p(x_i)$

'Entropy' $= -\sum p_i \log_2 p_i$

<u>Game</u>: give you a die

$A = \{1,2,3\}$, $B = \{4,5\}$, $C = \{6\}$

If A occurs then you gain 1 rupee

B . . . . 2 rupee

C . . . . 3 rupee.

Q. How do I set an entry fee to the game?

$S = \{1, 2, 3\}$

Q. On average, how much I have to pay?

$p(A) = \frac{3}{6}$, $p(B) = \frac{2}{6}$, $p(C) = \frac{1}{6}$.

expected amount of pay

$= \left[(1) \times \left(\frac{3}{6}\right)\right] + \left[2 \times \frac{2}{6}\right] + \left[3 \times \frac{1}{6}\right]$

$= \frac{3}{6} + \frac{4}{6} + \frac{3}{6} = \frac{10}{6} = \frac{5}{3}$.

conclusion is I will loose if 1 rupee is set as the entry fee.

Obs. If I set 2 rupee as the entry fee, I expect to gain

$2 - \frac{5}{3} = \frac{1}{3}$ rupee for each play.

Expectation:

$$E(X) = \sum_{x \in S_x} x f(x), \quad f(x) = P(X = x).$$

Consider a new random variable

$$Y = X^2$$

<u>pmf of Y</u>

$$g(y) = \begin{cases} \frac{4-\sqrt{y}}{6}, & y = 1, 4, 9 \\ 0, & \text{otherwise} \end{cases}$$

$$f(x) = P(X = x)$$

$$= \begin{cases} \frac{4-x}{6}, & x = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

pmf corresponding to previous r.v.

<u>Q</u>. $E(Y) = ?$

$$= \sum_{y=1,4,9} y\, g(y)$$

$$= \frac{10}{3}$$

$$E(Y) = \sum_{x=1,2,3} x^2 f(x) = \frac{10}{3}.$$

<u>Defn.</u> Let $f(x)$ be the pmf of a r.v. X. Then

$$E(u(X)) = \sum_{x \in S_X} u(x) f(x)$$

where $u(X)$ is a fn of X.

Quiz:

Find $E(X)$, where X is the r.v. corresponding to hypergeometric distribution.

Ans: $E(X) = n \frac{N_1}{N}$. (H.W.).

Another distribution.

An experiment has probability of success p, $0 < p < 1$, and probability of failure $= 1-p = q$.

$S_X =$
The experiment is repeated independently until the first success occurs, say this happens on the X trial.

$S_X = \{1, 2, 3, 4, \dots\}$

Then

$$f(x) = P(X = x)$$

$$= \underbrace{q \cdot q \cdots q}_{(x-1) \text{ times}} \, p$$

geometric distribution (pmf)

$$= q^{x-1} p, \quad x \in S_X$$

Q. 1. $\sum_{x \in S_X} f(x) = 1$? ok

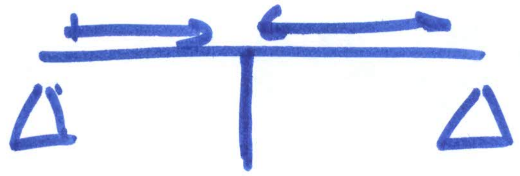
2. $E(X) = ?$

<u>Ans.</u> $= \frac{1}{1-q} = \frac{1}{p}$.

We shall denote

$$E(X) \equiv \mu_X \text{ or } \mu.$$

Recall Classical mechanics.



moment = force × moment arm.

$$S = \{u_1, u_2, \ldots u_k\}$$

$$p(X = u_i) = f(u_i) > 0$$

$$\sum_{x \in S} f(x) = 1$$

$$\text{mean} = \sum_{x \in S} x f x$$

$$= u_1 f(u_1) + \cdots + u_k f(u_k)$$

u_i — distance of mass ith pt. from the origin / center.

$u_i f(u_i)$ — moment having a moment arm of length u_i.

The sum of products = the moment of the system of distances and weights (gravitational force × mass).

Let us compute the first moment about the mean μ. Then

$$\sum_{x \in S} (x-\mu) f(x)$$

$$= E[(X-\mu)]$$

$$= E(X) - \mu \quad ?$$

$$= \mu - \mu = 0$$

<u>H.W.</u> Find those pts. x in the previous example of the 'Game' for which it will give positive or negative moments.

~~The weighted mean of~~

The second mont is

$$E[(X-\mu)^2] = \sum_{x \in S_X} (x-\mu)^2 f(x)$$

$= \sigma^2$, called the variance correspondig to the r.v. X.

The square root of variance is called the standard deviation.

$$\sigma^2 = E[(X-\mu)^2]$$

$$= E[X^2 - 2X\mu + \mu^2]$$

$$= E(X^2) - 2\mu E(X) + \mu^2$$

$$= E(X^2) \underbrace{- 2\mu E(X) + 2\mu^2}_{0} - \mu^2$$

$$= E(X^2) - \mu^2.$$

If $Y = aX + b$

$\sigma_y^2 = a^2 \sigma_x^2$, $\sigma_y = |a| \sigma_x$.

For any positive integer 'r', r-th moment of the r.v. about the mean is

$$E[(X-\mu)^r] = \sum (x-\mu)^r f(x)$$

Exp.

$$\mu = E(X) = n\left(\frac{N_1}{N}\right) = n'p'.$$

$$\sigma^2 = np(1-p)\frac{N-n}{N-1}.$$

Properties of Expectation: 'E' is a linear fn.

1. $E(c) = c$
2. $E(c\,u(X)) = c\,E[u(X)]$
3. $E[c\,u_1(X) + d\,u_2(X)] = c\,E[u_1(X)] + d\,E[u_2(X)]$