Probability & Statistics.

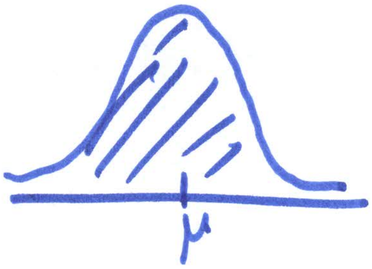
Lec-10

Normal distribution.

pdf: $$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \quad -\infty < x < \infty$$

$\sigma > 0$ and $-\infty < \mu < \infty$.

$X \equiv N(\mu, \sigma^2)$



MGF of X:

$$M(t) = \int_{-\infty}^{\infty} \frac{e^{tx}}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}\left[x^2 - 2(\mu+\sigma^2 t)x + \mu^2\right]\right\} dx$$

$$x^2 - 2(\mu+\sigma^2 t)x + \mu^2 = \left[x - (\mu+\sigma^2 t)\right]^2 - 2\mu\sigma^2 t - \sigma^4 t^2$$

$$= \exp\left(\frac{2\mu\sigma^2 t + \sigma^4 t^2}{2\sigma^2}\right) \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}\left[x - (\mu+\sigma^2 t)\right]^2\right] dx$$

$$= \exp \frac{2\mu\sigma^2 t + \sigma^4 t^2}{2\sigma^2} = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right]$$

$$E(X) = M'(0) = \mu$$

$$\text{var}(X) = M''(0) - [M'(0)]^2$$

$$= \sigma^2.$$

If $Y = N(0,1)$ it is called standard normal distribution. and the cdf of Y is

$$\Phi(y) = P(Y \le y)$$

$$= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw.$$

Exp. If $Z \approx N(0,1)$, find

$$P(Z \le 1.24) = \Phi(1.24) = 0.8925$$

$$P(1.24 \le Z \le 2.37) = \Phi(2.37) - \Phi(1.24).$$

$$P(-2.37 \le Z \le -1.24)$$

Thm. If $X = N(\mu, \sigma^2)$ then

$Z = \frac{X-\mu}{\sigma}$ then $Z \approx N(0,1)$

Pf. cdf. of Z

$$P(Z \le z) = P\left(\frac{X-\mu}{\sigma} \le z\right)$$

$$= P(X \le \sigma z + \mu)$$

$$= \int_{-\infty}^{\sigma z+\mu} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx$$

Let $w = \frac{x-\mu}{\sigma}$

$$= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} \, dw.$$

✓ which is $N(0,1)$

If $X \approx N(\mu, \sigma^2)$

$$P(a \leq X \leq b) = P\left(\frac{a-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Exp. If $X \approx N(25, 36)$, find 'c' s.t.

$$P(|X-25| \leq c) = 0.9544.$$

Ans. 12.

Q. If $X \approx N(\mu, \sigma^2)$ then

let $Z^2 = \frac{(X-\mu)^2}{\sigma^2}$

Find the distribution of Z^2.

<u>Sol</u>: Let $V = Z^2$

where $Z = \frac{X-\mu}{\sigma}$.

<u>cdf of V</u>

$$G(v) = P(V \le v)$$
$$= P(Z^2 \le v)$$
$$= P(-\sqrt{v} \le Z \le \sqrt{v})$$
$$= \int_{-\sqrt{v}}^{\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$
$$= 2\int_{0}^{\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

Let $z = \sqrt{y}$

i.e. $\frac{d}{dy}(\sqrt{y}) = \frac{1}{2\sqrt{y}}$

$$\Rightarrow \quad G(v) = \int_0^v \frac{1}{\sqrt{2\pi y}} e^{-y/2} dy.$$

Then pdf is $g(v) = G'(v) = \frac{1}{\sqrt{\pi}\sqrt{2}} v^{1/2-1} e^{-v/2}$.

$\longrightarrow \chi^2(1)$

Functions of random variables.

Exp. Let X have Gamma distribution.

$$f(x) = \frac{1}{\Gamma(\alpha)\,\theta^{\alpha}}\, x^{\alpha-1} e^{-x/\theta}, \quad 0 < x < \infty$$

where $\alpha > 0$, $\theta > 0$.

Find the pdf of $Y = e^{X}$.

Called **loggamma**.

Exp. Let W be the uniform distribution on the interval $(-\pi/2, \pi/2)$.

Then find the distribution of the r.v. $X = \tan W$.

Cauchy distribution