MA 20205 Probability and Statistics
Assignment No. 6

1. Let (X, Y) have the joint pdf $f_{X,Y}(x,y) = e^{-(x+y)}, x > 0, y > 0$. Find $P(1 < X + Y < 2), P(X < Y | X < 2Y), P(0 < X < 1 | Y = 2)$. Determine m such that $(X + Y < m) = \frac{1}{2}$.
2. Let (X, Y) have the joint pdf $f_{X,Y}(x,y) = x + y,\ 0 < x < 1, 0 < y < 1$. Are X and Y independent? Find $V(X + Y), Corr(X, Y)$ and $Var(X | Y = y)$.
3. Let (X, Y) have the joint pdf

$$f_{X,Y}(x,y) = \frac{1}{8}(6 - x - y), 0 < x < 2, 2 < y < 4.$$

   Find $E(Y|X = x), Var(Y|X = x), Cov(X, Y), Var(X|Y = y)$.
4. The ages (in years) at marriage of women and men in a country can be modelled by a BVN (24, 28, 36, 49, 0.8) distribution (the first variable corresponds to age of wife and the second to the age of her husband). Find the proportion of women over 30. What is variance of age of wives whose husbands are of age 35? Also find the proportion of women over 30 having husbands of age 35. What is the expected age of men with wives of age 22.

5. The life of a tube X_1 and the filament diameter X_2 are distributed as a bivariate normal with $\mu_1 = 1000$ hrs, $\mu_2 = 0.1$ cm, $\sigma_1 = 20, \sigma_2 = 0.1, \rho = 0.75$. A quality control manager wishes to determine the life of each tube by measuring the filament diameter. If a filament diameter is 0.09 cm, what is the probability that the tube will last 980 hrs?

6. Let (X, Y) be continuous with the joint pdf given by

$$f_{X,Y}(x,y) = \frac{1}{y} Exp\left\{-\left(y + \frac{x}{y}\right)\right\}, x > 0, y > 0.$$

   Find the marginal density of Y and the conditional density of $X|Y = y$. Hence, or otherwise, evaluate $E(Y), E(X), Var(Y), Var(X), Cov(X, Y)$ and $Corr(X, Y)$.
7. A professor has noticed that marks on two successive tests have a bivariate normal distribution with $\mu_1 = 75, \mu_2 = 83, \sigma_1 = 5, \sigma_2 = 4$ and $\rho = 0.8$. If a student received a grade of 80 on the first test, what is the probability that he/she will do better on the second one? How is the answer affected by taking $\rho = -0.8$?

8. The amount of rainfall recorded (in inches) at a weather station in April is a random variable X_1 and the amount of rainfall recorded at the same station in May is a random variable X_2. Let (X_1, X_2) have a bivariate normal distribution with parameters $\mu_1 = 6, \mu_2 = 4, \sigma_1 = 1, \sigma_2 = 0.5, \rho = 0.1$. Find $P(X_1 \le 5)$, $P(X_2 \le 5 | X_1 = 5)$ and $E(X_1 | X_2 = 6)$.

9. Let X and Y have joint density function

$$f_{X,Y}(x,y) = 1 - \alpha(1-2x)(1-2y), \quad 0 < x < 1, 0 < y < 1$$

Determine the valid range of α. Find correlation between X and Y. Show that X and Y are independent if and only if they are uncorrelated.

10. Let X denote the number of 'do loops' in a FORTRAN programme and Y denote the number of runs needed for a novice to debug the programme. Assume that the joint pdf of (X,Y) is given by

$X \setminus Y$	1	2	3	4
0	.059	.100	.050	.001
1	.093	.120	.082	.003
2	.065	.102	.100	.010
3	.050	.075	.070	.020

a. Find the probability that a randomly selected programme contains at most one 'do loop' and requires at least two runs to debug the programme.

b. Find marginal pmf's of X and Y Calculate $\rho_{X,Y}$.

c. Find the probability that a randomly selected programme requires at least two runs to debug given that it contains exactly one 'do loop'.

11. An electronics device is designed to switch house lights on and off at random times after it has been activated and assume that it is designed in such a way that it will be switched on and off exactly once in a one-hour period. Let Y denote the time at which lights are turned on and X the time at which they are turned off. Assume that the joint density of X and Y is given by

$$f_{X,Y}(x,y) = 8xy, \qquad 0 < y < x < 1.$$

(a) Find the probability that the lights will be switched on within 1/2 hour after being activated and then switched off again within 15 minutes.

(b) Find the probability that the lights will be switched off within 45 minutes of the system being activated given that they were switched on 10 minutes after the system was activated.

(c) Find the expected time that the lights will be turned off given they were turned on 10 minutes after activation.

(d) Find coefficient of correlation between X and Y.

12. Let (X,Y) be continuous with the joint probability density function

$$f_{X,Y}(x,y) = \frac{1}{4}(1+xy), |x| < 1, |y| < 1 .$$

Find $P(2X < Y), P(|X+Y| < 1)$ and $\rho_{X,Y}$.

13. Let (X, Y) have joint pdf

$$f_{X,Y}(x, y) = c, \;\; x^2 \le y < x, 0 < x < 1.$$

Find the value of c. Also determine $\rho_{X,Y}$ and $P\left(\frac{1}{3} < X < \frac{2}{3}\right)$, $P\left(\frac{1}{4} < Y < \frac{3}{4}\right)$, $P\left(\frac{5}{16} < X < \frac{7}{16} | Y = \frac{1}{4}\right)$, $P\left(\frac{3}{8} < Y < \frac{5}{8} | X = \frac{1}{2}\right)$ and $P(|X - Y| < 0.5)$.

14. Let (X_1, X_2) have joint pdf

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2}(x_1 + x_2)\, e^{-(x_1+x_2)} \;, \;\; x_1 > 0, x_2 > 0.$$

Find $E(X_2|X_1 = 4)$, $Var(X_1|X_2 = 2)$ and ρ_{X_1,X_2}.

15. Let (X, Y) have bivariate normal distribution with parameters $(-1, 1, 4, 9, -0.5)$. Find $E\{(X + 1)^2(Y - 1)^2\}$, $P(X < 1)$, $P(Y > 4)$, $P\left(-\frac{3}{2} < X < -\frac{1}{2} | Y = \frac{3}{2}\right)$ and $P\left(\frac{1}{2} < Y < \frac{3}{2} | X = -\frac{1}{2}\right)$.

16. Let (X, Y) have bivariate normal distribution with density function

$$f_{X,Y}(x, y) = \frac{3}{4\pi\sqrt{2}}\, Exp\left\{-\frac{9}{16}\left[(x - 1)^2 - \frac{2}{3}(x - 1)(y - 1) + (y - 1)^2\right]\right\},$$

$$-\infty < x < \infty, -\infty < y < \infty.$$

Find $P(4 < 2X + 3Y < 6), E(X|Y = 2)$ and $Var(Y|X = 0)$.

17. Let (X, Y) be jointly distributed continuous random variables with pdf

$$f_{X,Y}(x, y) = \frac{2x+y}{4}, \;\; 0 < x < 1, 0 < y < 2.$$

Find marginal and conditional distributions of X and Y. Also determine $P(|Y - X| > 1)$ and $\rho_{X,Y}$.