MA20104 Probability and Statistics

Problem Set 5

1. Let X be the sine of an angle in radians chosen uniformly from $(-\pi/2, \pi/2)$. Find the mean and variance of X.

2. Let X have the normal density $n(0, \sigma^2)$. Find the mean and variance of each of the following random variables: (a) $|X|$; (b) X^2 ; (c) e^{tX} .

3. Let X be a nonnegative continuous random variable having density f and distribution function F. Show that X has finite expectation if and only if

$$\int_0^\infty (1 - F(X))dx < \infty$$

and then

$$EX = \int_0^\infty (1 - F(X))dx$$

4. Let X and Y be independent random variables such that X has the normal density $n(\mu, \sigma^2)$ and Y has the gamma density $\Gamma(\alpha, \lambda)$. Find the mean and variance of the random variable $Z = XY$.

5. Let X, Y, and Z be random variables having mean 0 and unit variance. Let ρ_1 be the correlation between X and Y, ρ_2 the correlation between Y and Z, and ρ_3 the correlation between X and Z. Show that

$$\rho_3 \geq \rho_1\rho_2 - \sqrt{1-\rho_1^2}\sqrt{1-\rho_2^2}$$

6. Let $X_1, X_2, ...$ be independent, identically distributed random variables having mean 0 and finite nonzero variance σ^2 and set $S_n = X_1 + X_2 + \cdots + X_n$. Show that if X_l has finite third moment, then $ES_n^3 = nEX_n^3$ and $\lim_{n\to\infty} E(\frac{S_n}{\sigma\sqrt{n}})^3 = 0$, which is the third moment of the standard normal distribution.

7. Let $X_1, X_2, ...$ and S_n be as in the previous problem. Show that if X_l has finite fourth moment, then $ES_n^4 = nEX_1^4 + 3n(n-1)\sigma^4$ and $\lim_{n\to\infty} E(\frac{S_n}{\sigma\sqrt{n}})^4 = 3$, which is the fourth moment of the standard normal distribution. Hint: The term 3n(n 1) comes from the expression $n_{C_2}\frac{4!}{2!2!}$

8. Let $X_1, X_2, \ldots$ be independent normally distributed random variables having mean 0 and variance 1 (a) Find $P(X_1^2 + \ + X_1 00^2 \leq 120)$. (b) Find $P(80 \leq X_1^2 + \ + X_1 00^2 \leq 120)$. (c) Find c such that $P(X_1^2 + \ + X_1 00^2 \leq 100 + c) = 0.95$. (d) Find c such that $P(100 - c \leq X_1^2 + \ + X_1 00^2 \leq 100 + c) = 0.95$.

9. A runner attempts to pace off 100 meters for an informal race. His paces are independently distributed with mean $\mu = 0.97$ meters and standard deviation $\sigma = 0.1$ meter. Find the probability that his 100 paces will differ from 100 meters by no more than 5 meters.

10. Twenty numbers are rounded off to the nearest integer and then added. Assume the individual round-off errors are independent and uniformly distributed over $(-1/2, 1/2)$. Find the probability that the given sum will differ from the sum of the original twenty numbers by more than 3.

11. Let S_n have a binomial distribution with parameters n and $p = 1/2$. How does $P(S_{2n} = n)$ behave for n large?

12. Let X_1, X_2, X_3 be independent random variables each uniformly distributed on $(0, 1)$. Find the density of the random variable $Y = X_1 + X_2 + X_3$. Find $P(X_1 + X_2 + X_3 \leq 2)$.

13. Let X_l be chosen uniformly on $(0, 1)$, let X_2 be chosen uniformly on $(0, X_1)$ and let X_3 be chosen uniformly on $(0, X_2)$. Find the joint density of X_1, X_2, X_3 and the marginal density of X_3 .

14. Let $U_1, \ldots, U_n$ be independent random variables each having an exponential density with parameter λ. Find the density of $X_1 = \min(U_1, \ldots, U_n)$.

15. Let X and Y be independent random variables each having density f. Find the joint density of X and $Z = X + Y$.

16. Let X and Y be independent random variables each having an exponential density with parameter λ. Find the conditional density of X given $X + Y = z$.

17. Let X and Y be positive continuous random variables having joint density f. Set $W = Y/X$ and $Z = X + Y$. Find the joint density of W and Z in terms of f.

As additional problems you may try the following problems from the fourth edition of the book "Probability and Statistics in Engineering" by the authors William Hines, Douglas Montgomery, David Goldsman, Connie M. Borror. The list of the problems is as follows:
Chapter 4: 4-3, 4-4, 4-5, 4-6, 4-9, 4-10, 4-14, 4-18, 4-19, 4-23, 4-23, 4-27, 4-28, 4-31, 4-35
Chapter 7: 7-19, 7-20, 7-21, 7-22, 7-23, 7-24, 7-25, 7-26, 7-27, 7-28, 7-29, 7-31, 7-38, 7-40, 7-41, 7-42, 7-43