MA20104 Probability and Statistics

Problem Set 3

1. Let N be a positive integer and let f be the function defined by

$$f(x) = \begin{cases} \frac{2x}{N(N+1)}, & x = 1, 2, \ldots, N, \\ 0, & \text{elsewhere.} \end{cases}$$

   Show that f is a discrete density and find its mean.

2. Let X be a random variable with $Binomial(n = 4, p)$. Find $E\left(sin(nX/2)\right)$.

3. Suppose X and Y are two random variables such that

$$P(|X - Y| < M) = 1$$

   for some constant M. Show that if Y has finite expectation, then X has finite expectation and $|EX - EY| < M$.

4. Let X and Y be two independent random variables having finite second moments. Compute the mean and variance of $2X + 3Y$ in terms of those of X and Y.

5. Suppose n balls are distributed at random into r boxes. Let $X_i = 1$ if box i is empty and let $X_i = 0$ otherwise.
   (a) Compute EX_j.
   (b) For $i \neq j$, compute $E(X_i X_j)$.
   (c) Let S_r denote the number of empty boxes. Write $S_r = X_1 + \cdots + X_r$, and hence compute $E(S_r)$.
   (d) Compute $Var(S_r)$.

6. Suppose we have two decks of n cards, each numbered $1, \ldots, n$. The two decks are shuffled and the cards are matched against each other. We say a match occurs at position i if the i^{th} card from each deck has the same number. Let S_n denote the number of matches. Then compute $E(S_n)$ and $Var(S_n)$.
   Hint: Let $X_i = 1$ if there is a match at position i and let $X_i = 0$ otherwise. Then $S_n = X_1 + \cdots + X_n$.

7. Suppose a box has 3 balls labeled 1, 2, and 3. Two balls are selected without replacement from the box. Let X be the number on the first ball and let Y be the number on the second ball. Compute $Cov(X, Y)$ and $\rho(X, Y)$.

As additional problems you may try the following problems from the fourth edition of the book "Probability and Statistics in Engineering" by the authors William Hines, Douglas Montgomery, David Goldsman, Connie M. Borror. The list of the problems is as follows:
Chapter 2: Exercises 2-1, 2-2, 2-7, 2-8, 2-9, 2-10, 2-16, 2-18, 2-19, 2-20
Chapter 3: Exercises 3-1, 3-4, 3-6, 3-23