Information and Coding theory — Lec-15.

DMC, Source seq. $\bar{X}^N = (X_1, \ldots, X_N)$

Output seq. $\bar{Y}^N = (Y_1, \ldots, Y_N)$

ⓐ $\underbrace{I(\bar{X}^N; \bar{Y}^N)}_{} \le \sum_{n=1}^{N} \underbrace{I(X_n; Y_n)}_{}$

(i) $\boxed{I(\bar{X}^N; \bar{Y}^N) \le NC}$

where 'C' is the channel capacity.

Pf. $\boxed{I(\bar{X}^N; \bar{Y}^N)} = \boxed{H(\bar{Y}^N)} - \boxed{H(\bar{Y}^N | \bar{X}^N)}$

$$H(\bar{Y}^N \mid \bar{X}^N) = \sum_{\bar{x},\bar{y}} P(\bar{x},\bar{y}) \log \frac{1}{P(\bar{y}\mid\bar{x})}$$

$$= \sum_{\bar{x},\bar{y}} Q(\bar{x})\, P(\bar{y}\mid\bar{x}) \log \frac{1}{P(\bar{y}\mid\bar{x})}$$

$$= \sum_{\bar{x},\bar{y}} Q(\bar{x})\, P(\bar{y}\mid\bar{x}) \left\{ \sum_{n=1}^{N} \log \frac{1}{P(y_n \mid x_n)} \right.$$

$$\bar{x} = (x_1, \ldots, x_N)$$

$$\bar{y} = (y_1, \ldots, y_N)$$

$$\bar{X}^N = (X_1, \ldots, \boxed{X_N})$$

$$X_i$$

$$A = \{a_1, \ldots, a_K\}$$

Hint: avg. of the sum is the sum of the Averages

$$\overset{?}{=}$$

$$= \sum_{n=1}^{N} H(Y_n \mid X_n)$$

$(\boxed{a_1}\; a_2)$ $(a_3\, a_4)$

$A = \{a_1, \dots, a_K\} \leftarrow X$

(Y_1, Y_2)

$Q(a_k)$

$Q \to$ pmf of X_i

$(\boxed{X_1}\; \boxed{X_2}) = \bar{X}^2$

$X_1 \sim X$

$X_2 \sim X$

$\bar{X}^N$

X_n

pmf

$P(X = x) =$

$\bar{X}^N = (X_1, \dots, X_N)$

$$p(\bar{X}^N = \bar{x})$$

$$= p\big((X_1, \dots, X_n) = (x_1, \dots, x_N)\big)$$

$$= p(X_1 = x_1, X_2 = x_2, \dots, X_N = x_N)$$

$$\boxed{\bar{X} = (X_1, \dots, X_n)}$$

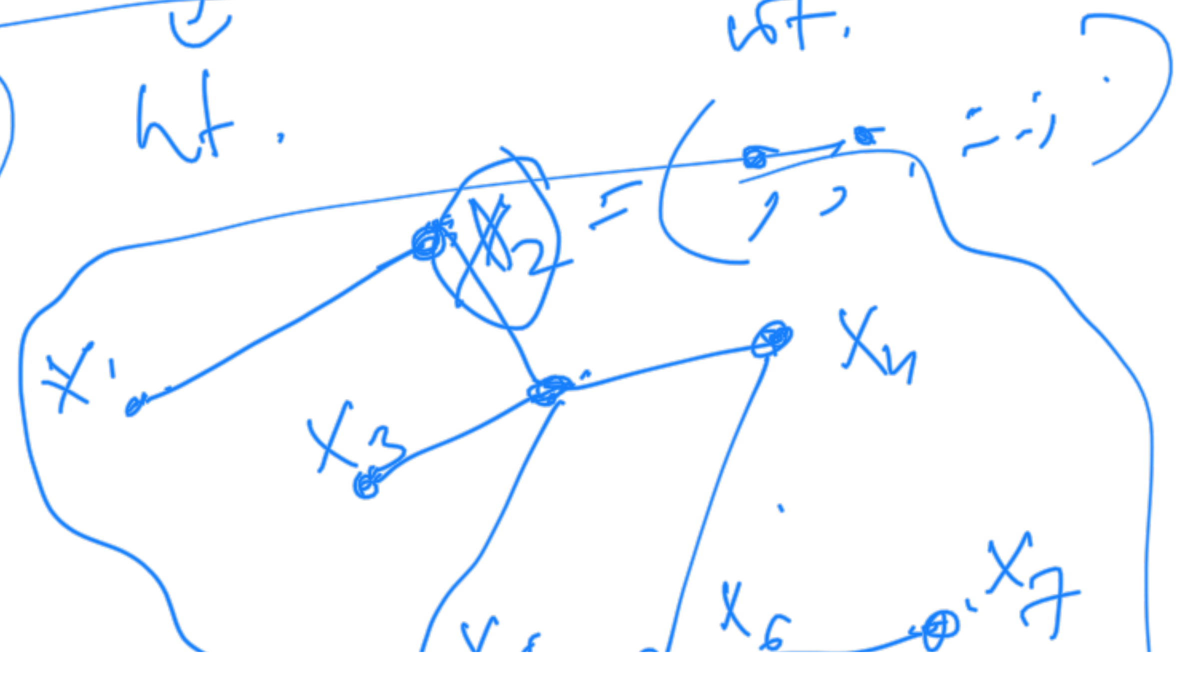
wt.

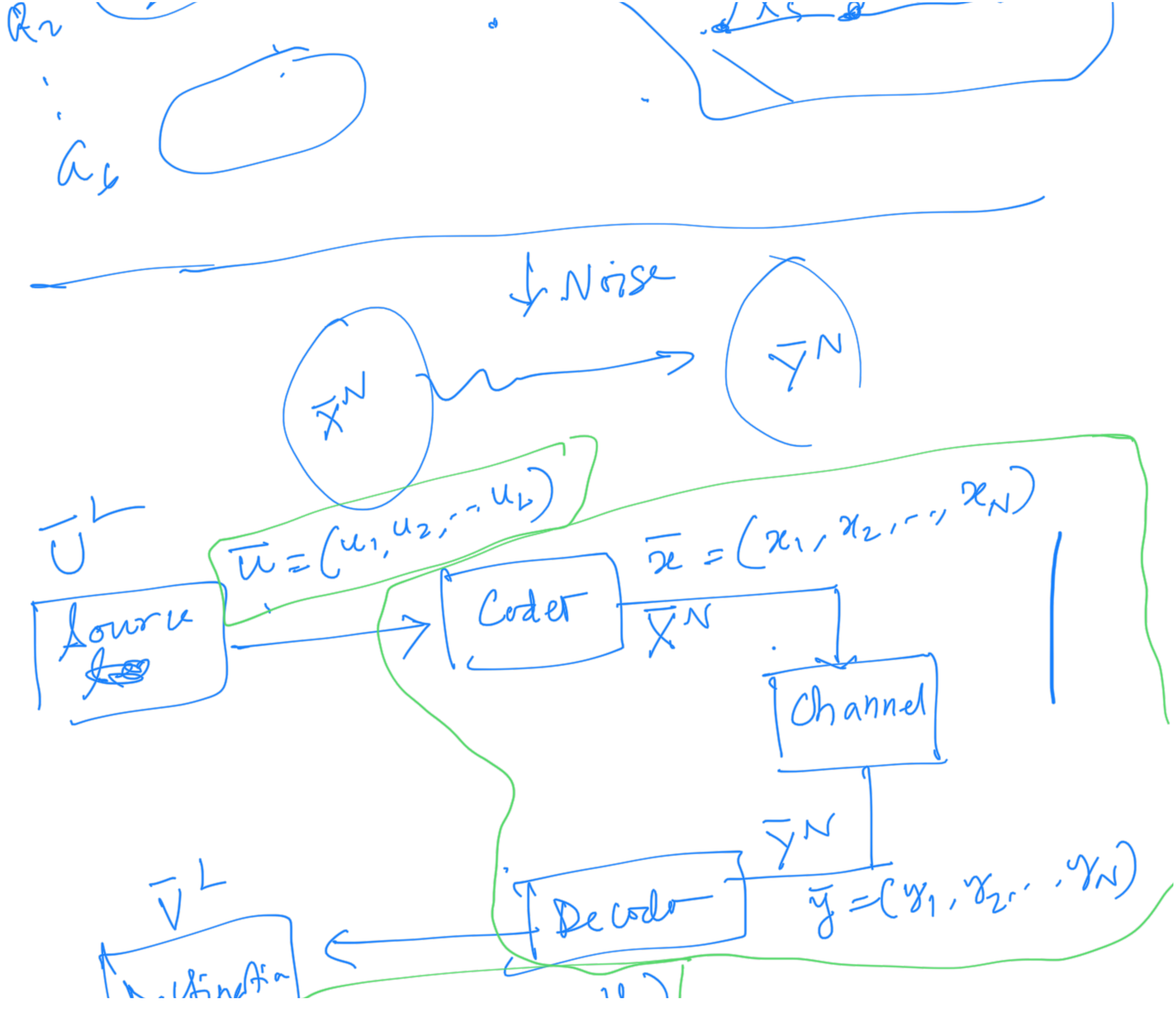
$I(\bar{X}_1 ; \bar{X}_2)$ wt.

$\mathbb{R}^N$

15 – 20

d_1 21 – 25





R_2
a_L
Noise
$\bar{X}^N$
$\bar{Y}^N$
$\bar{U}^L$
$\bar{u} = (u_1, u_2, \ldots, u_L)$
Source
Coder
$\bar{x} = (x_1, x_2, \ldots, x_N)$
$\bar{X}^N$
Channel
$\bar{Y}^N$
$\bar{y} = (y_1, y_2, \ldots, y_N)$
Decoder
$\bar{V}^L$

Define $\boxed{\bar{v} = (v_1, v_2, \dots, v_L)}$

$$a_1 a_2 a_3 \longrightarrow \text{"}\underline{010}\,\underline{100}\,\underline{111}\text{"}$$

Error: When $u_\ell \neq v_\ell$ for some $1 \le \ell \le L$

$$\boxed{P_{e,\ell} = p(\bar{u} \neq \bar{v}) = p(u_\ell \neq v_\ell)}$$

$\bar{u} = \bar{v}$

Average error probability.

$$\boxed{\langle P_e \rangle = \frac{1}{L} \sum_{\ell=1}^{L} P_{e,\ell}}$$??

Avg Ler

Result-1.

$$U \rightsquigarrow V$$

Sample space for both U & V. $\leftarrow$ $A = \{a_1, a_2, \dots, a_K\}$

$$\boxed{P_e = \sum_u \sum_{v \neq u} P(u,v)}$$

$P(u,v) \to$ pmf corresponding to (U,V)

Then,

$$P_e \log(K-1) + H(P_e) \geq H(U|V)$$

$$(1-p)\log(1-P_e)$$

$-p_e \log p_e - (1-p_e)$

Pf. $$H(U|V) = \sum_{u,v} p(u,v) \log \frac{1}{p(u|v)}$$

$$= \sum_{v} \sum_{u \neq v} p(u,v) \log \frac{1}{p(u|v)} + \sum_{v,\, u=v} p(u,v) \log \frac{1}{p(u|v)}$$

$$H(U|V) - p_e \log (K-1) - H(p_e) \overset{??}{=} \sum_{v} \sum_{u \neq v} p(u,v) \log \frac{p_e}{(K-1)\, p(u|v)} + \sum_{v,\, u=v} p(u,v) \log \frac{1-p_e}{p(u|v)}$$

$$\boxed{\log x \overset{?}{\leq} (\log e)(x-1)}$$??

x $\quad x$

$\log\left(\frac{1-p_e}{\ } - 1\right)$

RHS.

$$\le (\log e)\left[\sum_{v}\sum_{u\neq v} p(u,v)\left(\frac{p_e}{(K-1)\,p(u|v)} - 1\right) + \sum_{v,u=v} p(u,v)\left(\frac{\ldots}{p(u|v)} - \ldots\right)\right]$$

$$= (\log e)\left[\frac{p_e}{K-1}\sum_{v}\sum_{u\neq v} p(v) - \sum_{v}\sum_{u\neq v} p(u,v) + (1-p_e)\sum_{v} p(v) - \sum_{v,u=v} p(u,v)\right]$$

?? ?

$$\frac{p_e}{K-1} - p_e + (1-p_e) - (1-p_e)$$

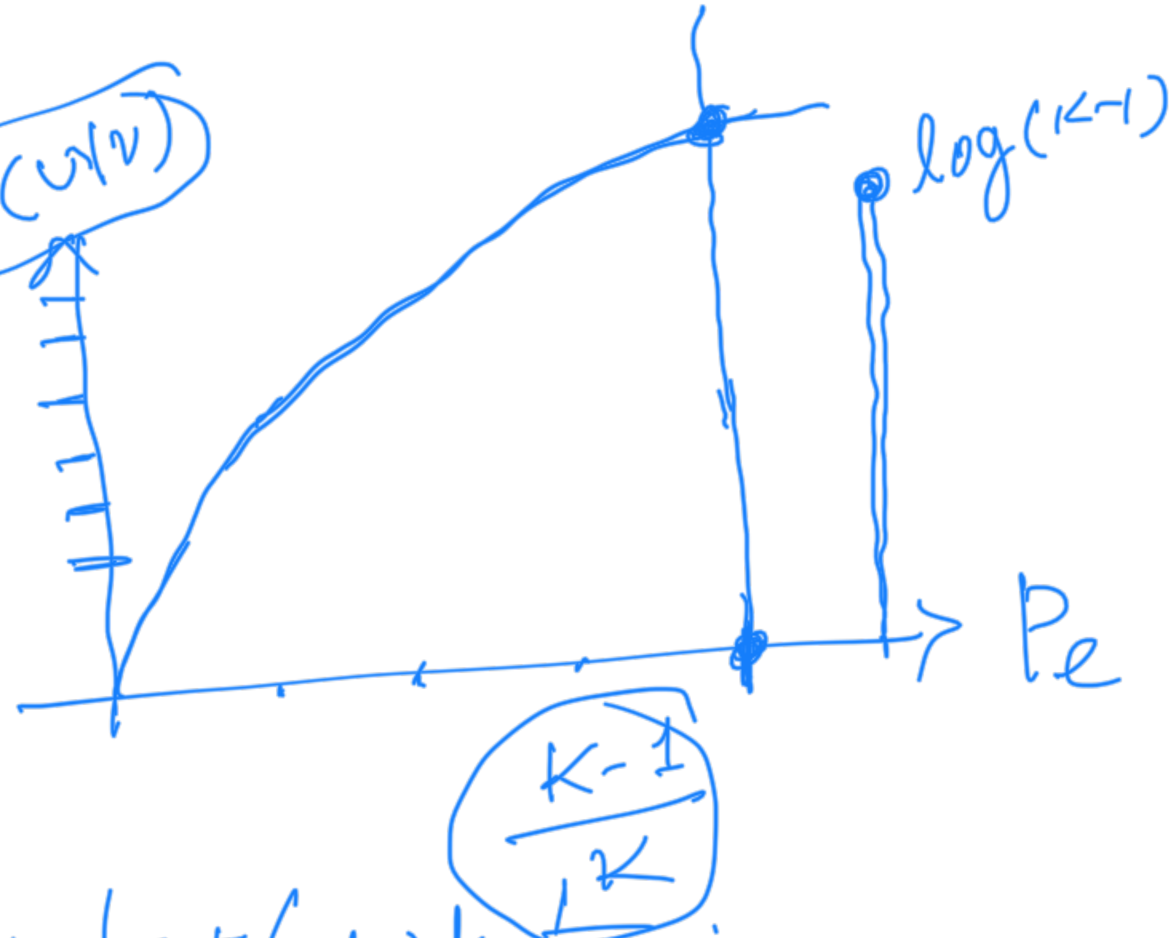
?

$$= \log_2 e\left[\frac{p_e}{K-1} + (1-p_e) - 1\right] \le 0$$

$$H(U|V) \le \underbrace{\boxed{p_e \log(K-1)}}_{} + \underbrace{H(P_e)}_{}$$

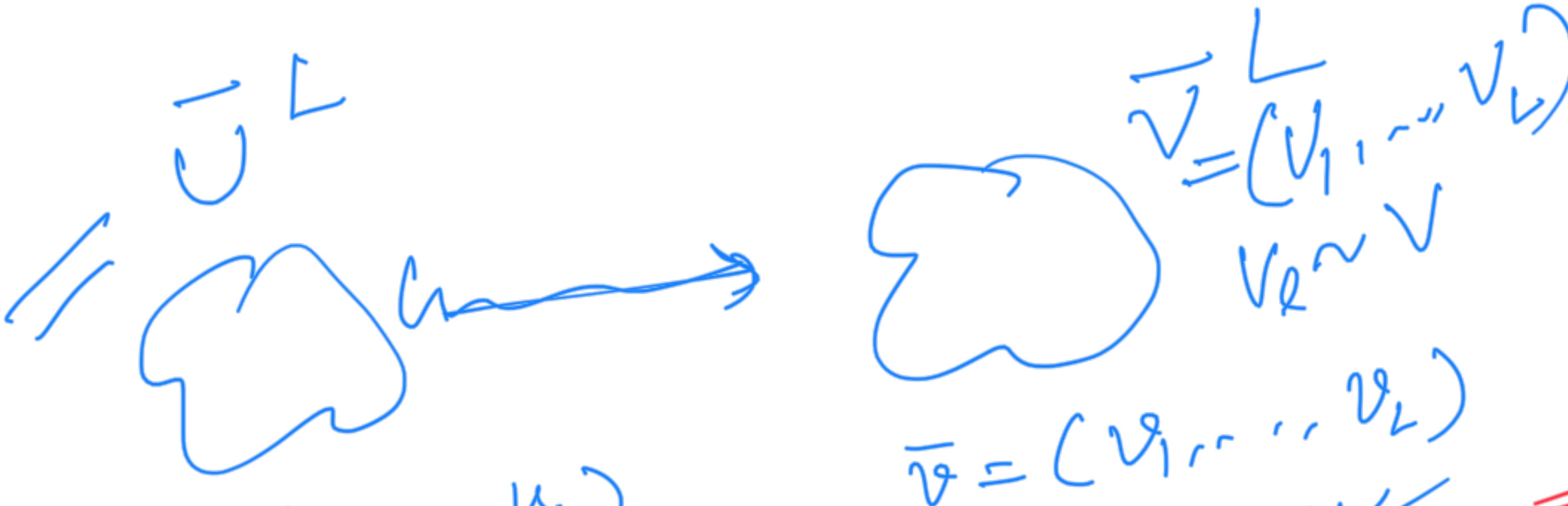
$\|$

$H(U) - I(U;V)$ $H(U|V)$



$$H(P_e) = P_e \log \frac{1}{P_e} + (1-P_e) \log \frac{1}{1-P_e}$$

Result.



$(U_1, U_2, \dots, U_L)$

$u_\ell \in \{a_1, \dots, a_K\}$

$\le L(\dots)$

$$v_l \sim$$

$$\langle p_e \rangle \log(K-1) + H(\langle p_e \rangle) \geq \frac{1}{L} H(U^L | V^L)$$

$$\geq H(U|V)$$

Pf. use: $P_e \log(K-1) + H(P_e) \geq H((U_1, U_2, \ldots, U_L) | \bar{V}^L)$

$$H(U^L | \bar{V}^L) = H((U_1, U_2, \ldots, U_L) | \bar{V}^L)$$

$X | Y$

$$= H(U_1 | \bar{V}^L) + H(U_2 | U_1 \bar{V}^L) + \cdots + H(U_L | U_1 \cdots U_{L-1} \bar{V}^L)$$

$$H(X_1, X_2, \ldots, X_n | Y)$$

$$\overset{?}{\leq} \sum_{l=1}^{L} H(U_l | V_l)$$

$$\leq P_{e,l} \log(K-1) + H(P_{e,l})$$

$$\underbrace{H(U_l | V_l)}$$

$$\underbrace{H(\bar{U}^L | \bar{V}^L)} \le \sum_{l=1}^{L} \left[P_{e,l} \log(K-1) + H(P_{e,l}) \right]$$

?

$$\Rightarrow \frac{1}{L} H(\bar{U}^L | \bar{V}^L) \le \log(K-1) \left(\sum_{l=1}^{L} \frac{1}{L} P_{e,l} + \frac{1}{L} \sum_{l=1}^{L} H(P_{e,l}) \right)$$

$$= \underbrace{\log(K-1) \langle P_e \rangle} + \underbrace{\frac{1}{L} \sum_{l=1}^{L} H(P_{e,l})}_{\le H(\langle P_e \rangle)}$$

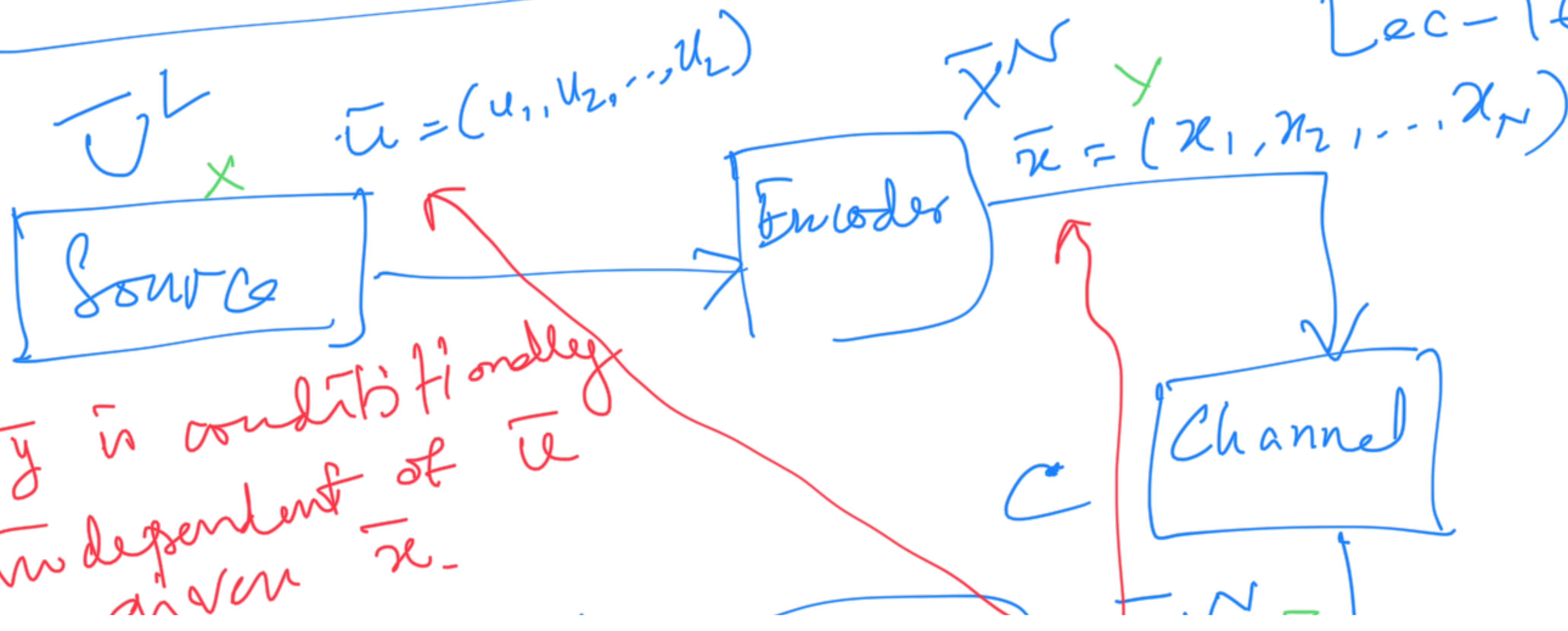
<u>Claim.</u> $\frac{1}{L} \sum^{L} H(P_{e,l}) \le H(\langle P_e \rangle)$

$$\sum_{l=1}^{L}$$

$$= \frac{1}{2}\left(\sum_{l=1}^{L} P_{e,l}\log\frac{1}{P_{e,l}} + \sum_{l=1}^{L}(1-P_{e,l})\log\frac{1}{1-P_{e,l}}\right)$$

<u>Hint.</u> $\boxed{\log x \le (\log e)(x-1)}$

Lec-16



$\bar{y}$ is conditionally independent of $\bar{u}$ given $\bar{x}$.

$\bar{v}$ is conditionally independent of $\bar{u}$ and $\bar{x}$ given $\bar{y}$.

For some l, $u_l \neq v_l \Rightarrow \bar{u} \neq \bar{v}$

Prob of error $\leftarrow$ $P_{e,l} = \sum_{\bar{u}_l \neq \bar{v}_l} p(\bar{u}, \bar{v})$

Avg. Prob. of error $\rightarrow$ $\boxed{\langle P_e \rangle} = \frac{1}{L} \sum_{l=1}^{L} P_{e,l}$

$\checkmark$

<u>Markov chain.</u> Suppose we have X, Y and Z, random variables. Then $X \rightarrow Y \rightarrow Z$ i.e. X, Y, Z form a Markov chain if the conditional ... only on Y

distribution of Z depends
and is conditionally independent of X.

?? This is ~~equals~~ equivalent to

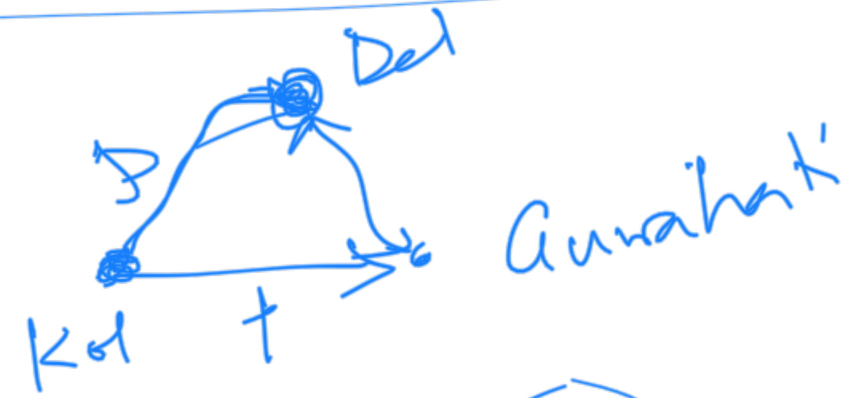
$$\rightarrow \underbrace{p(x, y, z)}_{\text{joint pmf of } (X, Y, Z)} = p(x)\, p(y|x)\, p(z|y)$$

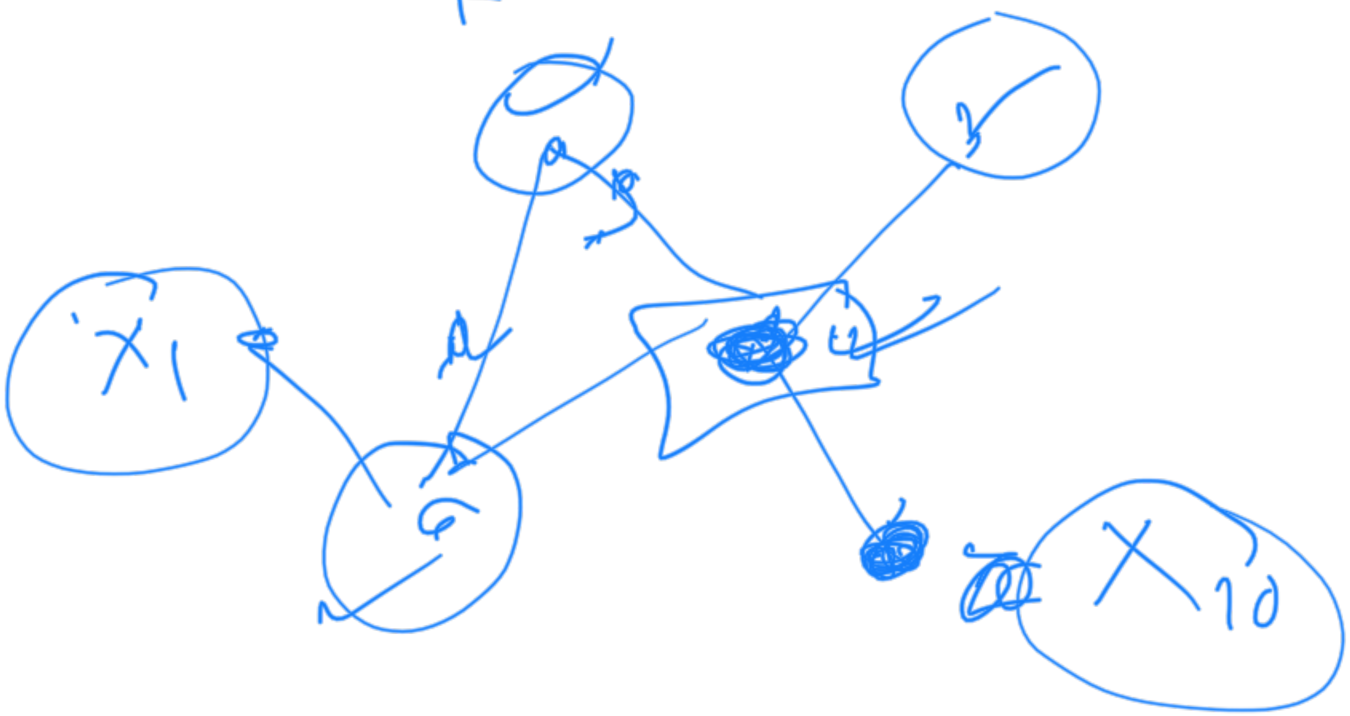
$\rightarrow X \rightarrow$ marks in CT-1
" $Y \rightarrow$ " " " CT-2
" $Z \rightarrow$ " " " CT-3

Say $\rightarrow \{ (10, 15, 25), \ldots \rightarrow (\ast, \ast, \ast) \} \leftarrow$ Sample space.

$$p(x_1, x_2, x_3) = \frac{1}{20 \times 20 \times 50}$$

$0 \leq x_1 \leq 20$
$0 \leq x_2 \leq 20$
$0 \leq x_3 \leq 50$
$\rightarrow$ Sample sp.





$$(x_1, \ldots, x_{10})$$

Result: Suppose $X \to Y \to Z$

$I(X;Y) \geq I(X;Z)$

Then $I(X;Y) \le$

Pf.

$$I(X;Y,Z) = I(X;Z) + \overbrace{I(X;Y|Z)}^{\geq 0}$$

$$= I(X;Y) + \underbrace{\boxed{I(X;Z|Y)}}_{0}$$

from the chain rule.

$$\Rightarrow \boxed{I(X;Z) \le I(X;Y)}$$

Similarly if $\boxed{X \to Z \to Y}$

$$I(Y;Z) \ge I(X;Y)$$

then

$$\boxed{I(Y;Z) \ge I(X;Z)}$$

H.W.

$$\bar{U}^L \longrightarrow \bar{X}^N \longrightarrow \underbrace{\bar{Y}^N}_{Z}$$

Markov chain

X

$$I(\bar{U}^L; \bar{Y}^N \mid \bar{X}^N) = 0]$$

(a) $I(\bar{U}^L; \bar{X}^N) \geq I(\bar{U}^L; \bar{Y}^N)$

(b) $I(\bar{X}^N; \bar{Y}^N) \geq I(\bar{U}^L; \bar{Y}^N)$

$$\underbrace{\bar{X}^N}_{X} \rightarrow \underbrace{\bar{Y}^N}_{Y} \rightarrow \underbrace{\bar{V}^L}_{Z}$$

$$\underset{X}{I(\overset{X}{\bar{X}^N}}; \underset{Z}{\bar{V}^L} \mid \underset{Y}{\bar{Y}^N}) = 0$$

(c) $I(\bar{X}^N; \bar{Y}^N) \geq I(\bar{X}^N; \bar{V}^L)$

(d) $I(\bar{Y}^N; \bar{V}^L) \geq I(\bar{X}^N; \bar{V}^L)$

Observation.

$$\bar{U}^L \to \bar{X}^N \to \bar{Y}^N \to \bar{V}^L$$

H.W.

$$\boxed{I(\bar{U}^L;\bar{V}^L)}\ \cancel{\geq}\ \boxed{\leq}\ \boxed{I(\bar{X}^N;\bar{Y}^N)}$$

Hint. $I(\bar{U}^L;\bar{V}^L \mid \bar{Y}^N) = 0$

Conclusion:

$$\langle P_e\rangle \log(K-1) + H(\langle P_e\rangle) \geq \frac{1}{L}\boxed{H(\bar{U}^L \mid \bar{V}^L)}$$

$$= \frac{1}{L}\left(H(\bar{U}^L) - I(\bar{U}^L;\bar{V}^L)\right)$$

$$= \frac{1}{L}\left(H(\bar{U}^L) - \dots\right)$$

$$\geq \frac{1}{L}\left(H(\bar{U}^L) - I(\bar{X}^N;\bar{Y}^N)\right)$$

$$\geq \frac{1}{L}H(\bar{U}^L) - \frac{N}{L}C$$

τ_s — time interval between two source letter

τ_c — time interval between two channel letter.

Assume. $N \approx \left\lfloor \frac{L\tau_s}{\tau_c} \right\rfloor$ i.e. $\boxed{N\tau_c = L\tau_s}$

Then,

$$\langle P_e \rangle \log (K-1) + H(\langle P_e \rangle) \geq \frac{1}{L} H(\bar{U}^L) - \frac{\tau_s}{\tau_c} C$$

inverse of Channel coding theorem.

Source rate

$> ?$

Channel capacity per source digit.