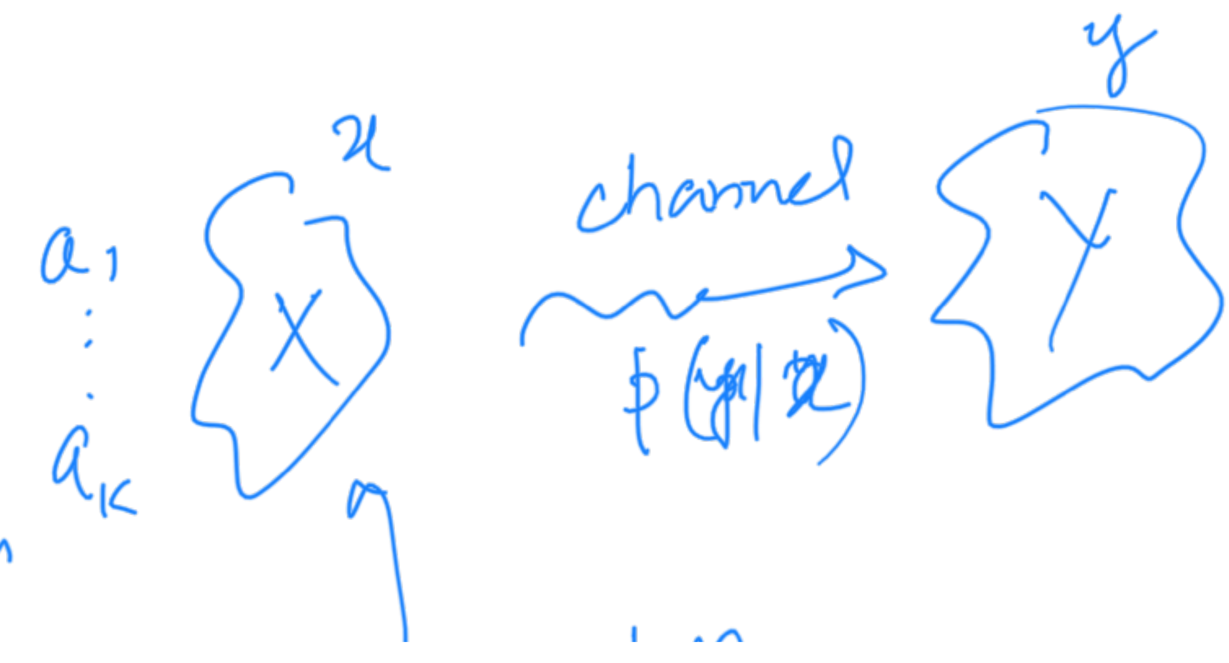
Information and coding theory.

Lec-13

Exam (class test) → On March 11, 2022 (Friday)

Code letter, $D = \{0, 1\}$



source letters

memoryless source.

Prefix code — Huffman code.

Entropy ~~⟶~~ length of codewords

Shannon-Fano Code.

$$A = \{a_1, a_2, \dots, a_K\}$$

$$n_1 \quad n_2 \quad \dots \quad n_K$$

$$\underbrace{p(a_1), \dots, p(a_K)}_{p(x)}$$

$$x \in A$$

$$\boxed{l(x) = \left\lceil \log \frac{1}{p(x)} \right\rceil}$$

$$A = \{1, 2, \dots, k\} \equiv \mathcal{X}$$

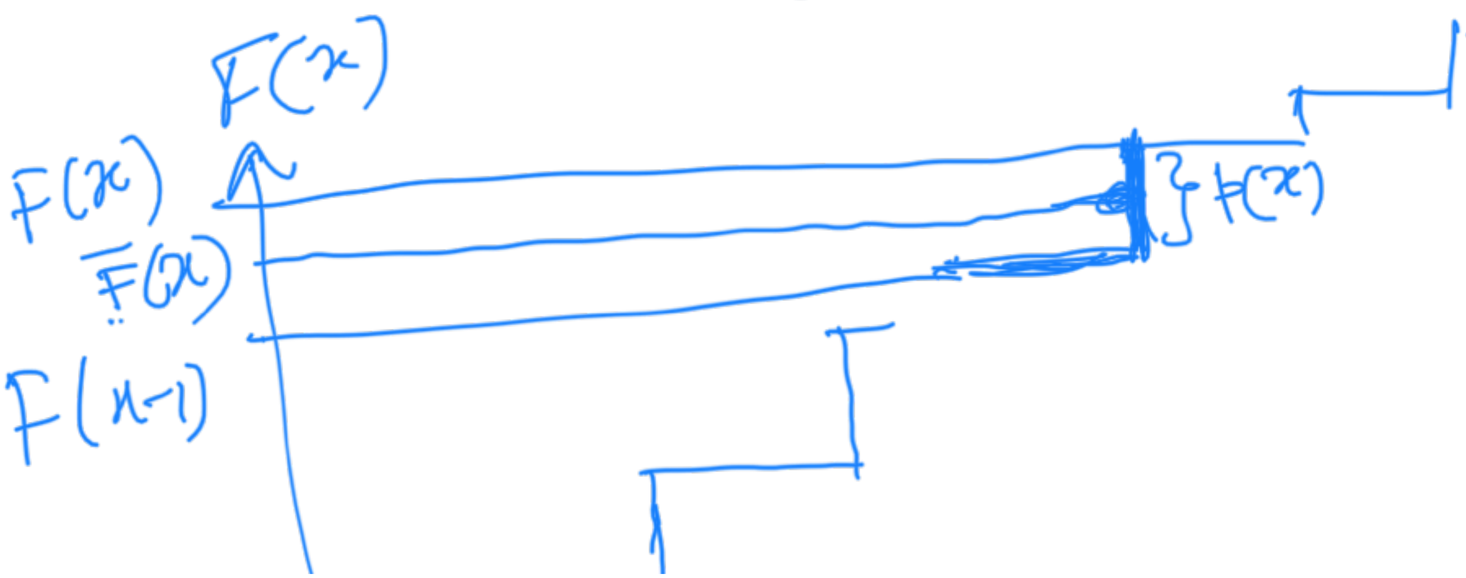
$$p(x) > 0$$

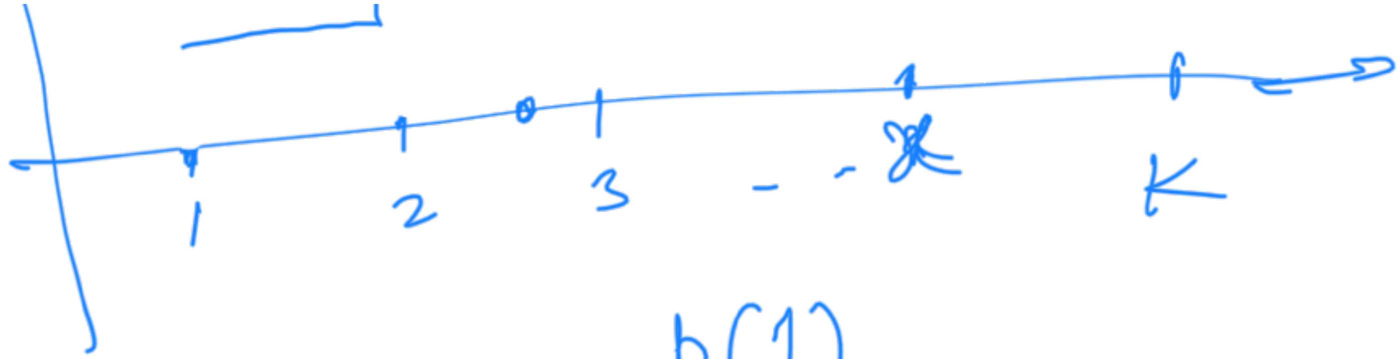
CDF: $$\boxed{F(x)} = \sum_{\boxed{a \le x}} p(a)$$

=

Modified CDF:

$$\bar{F}(x) = \underbrace{\sum_{a < x} p(a)}\ + \frac{1}{2}\, p(x)$$





$$F(1) = p(1)$$

$$F(2) = p(1) + p(2)$$

$$x \longrightarrow \boxed{p(x)},\ \boxed{F(x)},\ \boxed{F(x)}$$

$$A = \{0, 1\}, \qquad p(0) = \frac{1}{2} = \text{a seqn. of } 0, 1$$

$$p(1) = \frac{1}{2} = \text{a seq. of } 0, 1$$

$$\text{If } a_i \neq a_j \Rightarrow F(a_i) \neq F(a_j), \quad \boxed{p(x) > 0}$$

Possibility:

representation of $F(x)$.

binary

$$l(x) = \left\lceil \log \frac{1}{p(x)} \right\rceil$$

$x \in \{1, \ldots, k\}$.

$2^{\mathbb{N}} \sim |\mathbb{R}|$

$\downarrow$

$(0,1) \sim \mathbb{R}$

$\bar{F}(x) \longrightarrow$ discard all the bits in the binary representation of $\bar{F}(x)$, except the first $\boxed{l(x)}$ bits.

$= \lfloor \bar{F}(x) \rfloor_{l(x)}$

which number.

$0\,1\,0\,1\,1\,1\,0 = \ldots$

H.W.

$$\bar{F}(x) - \lfloor \bar{F}(x) \rfloor_{l(x)} < \frac{1}{2^{l(x)}}$$

but $\boxed{l(x) = \left\lceil \log \frac{1}{p(x)} \right\rceil + 1}$ bits

$$\frac{1}{2^{l(x)}} < \frac{p(x)}{2} = \underbrace{\bar{F}(x) - F(x-1)}$$

$$\bar{x}_l = x_1 x_2 \ldots x_l \rightarrow \text{codeword } (a_i)$$

$$\tilde{x}_1 \, \tilde{x}_2 \ldots \tilde{x}_{\tilde{l}} \rightarrow \text{codeword } (a_j)$$

$$\left[\cdot x_1 x_2 \ldots x_L,\ \cdot x_1 x_2 \ldots x_L + \frac{1}{2^l}\right)$$

Avg. length of the codewords

$$\text{Avg Len} = \sum_{x \in \mathcal{X}} p(x)\, l(x)$$

$$= \sum_{x \in \mathcal{X}} p(x) \left(\left\lceil \log \frac{1}{p(x)} \right\rceil + 1 \right)$$

$< \boxed{H(X) + 1}$?

$\boxed{H(X) + 2}$ ✓

H.W.

Syllabus for Class Test-1

$$\bar{F}(x) = \Big(\underbrace{\qquad\qquad}_{l(x)}\Big]\; \times \Big)$$

? Comparison betw Huffmann code & Shannon-Fano code ?

Channel Capacity.

$p(x)\ X \rightsquigarrow Y$

Def.

$$C \overset{?}{=} \max_{\boxed{p(x)}} \boxed{I(X;Y)}$$

$$A = \{a_1, \ldots, a_K\}$$

$$\geq 0 \quad \begin{bmatrix} p(a_1) \\ p(a_2) \\ \vdots \\ p(a_K) \end{bmatrix} \longleftarrow p(a_1) \ \ldots \ p(a_K)$$

constraint :-

$$\left\{ \begin{bmatrix} x_1 \\ \vdots \\ \vdots \end{bmatrix} \in \mathbb{R}^K \;\middle|\; x_i \geq 0, \sum x_i = 1 \right\}$$

det

x_K

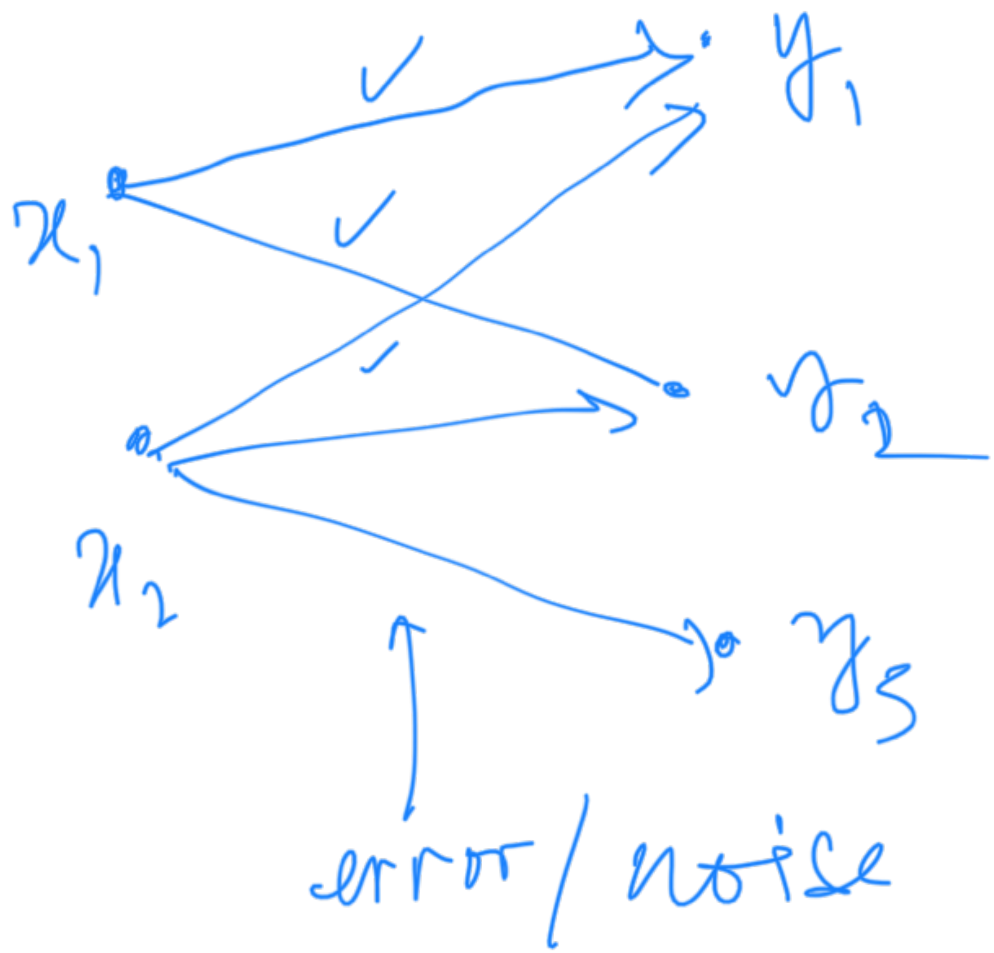
$\mathcal{H} : \{\text{pmf}\} \longrightarrow \mathbb{R}_{\geq 0}$

maximum rate

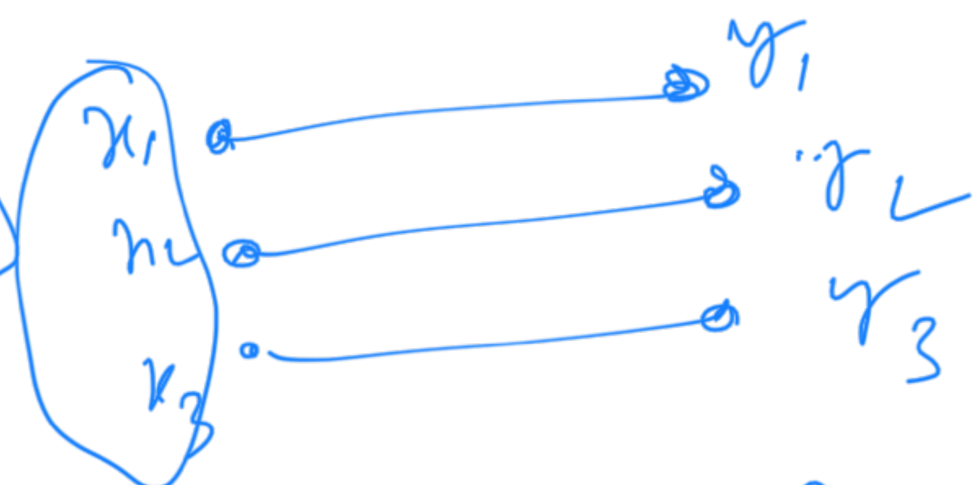
$C \longrightarrow$ maximum # of bits that can be transmitted through the channel (per use.)

Channel is described by the conditional probabilities $p(y|x)$

conditional



Noiseless channel.

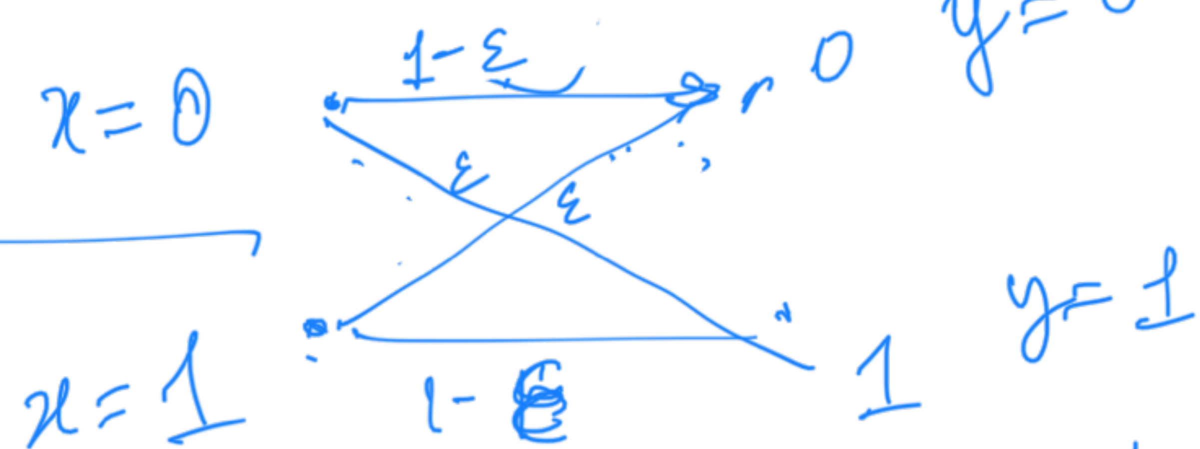


$C = \log_2 3$ bits

$I(X;Y)$

$$C = \max_{p(x)} I(X;Y) = H(Y) - H(Y|X)$$

$\varepsilon \to$ crossover probability.



Binary Symmetric Channel (BSC)

Channel capacity of BSC.

$p(x=0) = \alpha$

$$p(0) = p(\ldots$$

$$p(1) = p(x=1) = 1-u$$

$$I(X;Y) = \boxed{H(Y)} - \boxed{H(Y|X)}$$

$$p(y) = \sum_{x'=0}^{1} p(x')\, p(y|x')$$

$$= \left[p(x'=0)\, p(y|x'=0)\right] + \left[p(x'=1)\, p(y|x'=1)\right]$$

$$= \begin{cases} u(1-\varepsilon) + (1-u)\varepsilon & , y=0 \\ \ldots\ (1-u)(1-\varepsilon) & y=1 \end{cases}$$

$$(n\varepsilon + (1-n)(\ldots$$

$$H(Y|X) = \sum_{x=0}^{1} \sum_{y=0}^{1} p(x)\, p(y|x) \log_2 \frac{1}{p(y|x)}$$

$$= \sum_{x,y} p(x) \cdot H(Y|X=x)$$

$$= p(x=0)\, p(y=0|x=0) \log_2 \frac{1}{p(y=0|x=0)}$$

$$+ p(x=0)\, p(y=1|x=0) \log_2 \frac{1}{p(y=1|x=0)}$$

$$+ p(x=1)\, p(y=0|x=1) \log \frac{1}{p(y=0|x=1)}$$

$$+ p(x=1)\, p(y=1|x=1) \log \frac{1}{p(y=1|x=1)}$$

$$= u(1-\varepsilon)\log\frac{1}{1-\varepsilon} + u\varepsilon\log\frac{1}{\varepsilon} + (1-u)\varepsilon\log\frac{1}{\varepsilon} + (1-u)(1-\varepsilon)\log\frac{1}{1-\varepsilon}$$

$$= (1-\varepsilon)\log\frac{1}{1-\varepsilon} + \varepsilon\log\frac{1}{\varepsilon}$$

$$= \underline{\underline{H(\varepsilon)}}$$

$$I(X;Y) = \underbrace{H(Y)} - \underbrace{H(Y|X)}$$

$$= [u(1-\varepsilon) + (1-u)\varepsilon]\log\frac{1}{u\varepsilon + (1-u)(1-\varepsilon)} + [u\varepsilon + (1-u)(1-\varepsilon)]\log\frac{1}{u\varepsilon + (1-u)(1-\varepsilon)}$$

$H(\varepsilon)$

$$\max_{\begin{bmatrix} u \\ 1-u \end{bmatrix}} I(X;Y)$$

$$= \max_{u} \left[\text{'function } u\text{'} \right]$$

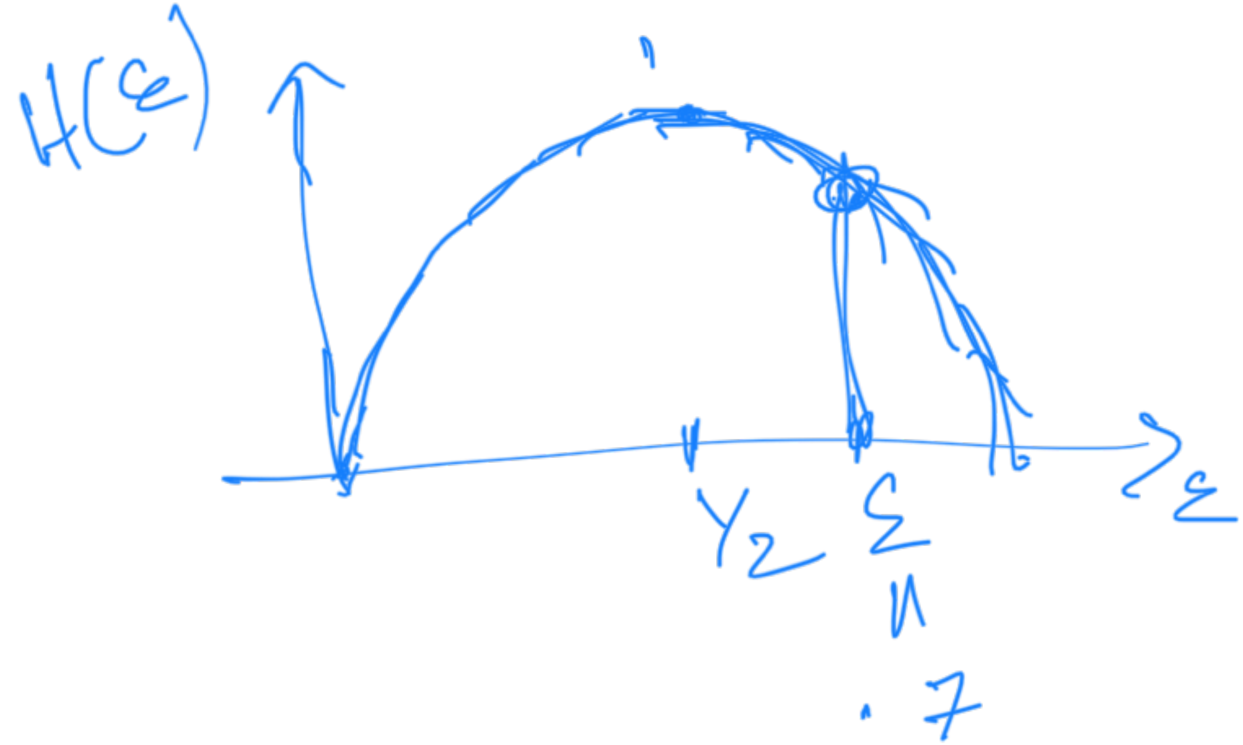
(H.W.) $\longrightarrow$ differentiate wrt. 'u' and calculate the value of u where it vanishes.

$$\boxed{u^* = \frac{1}{2}}$$

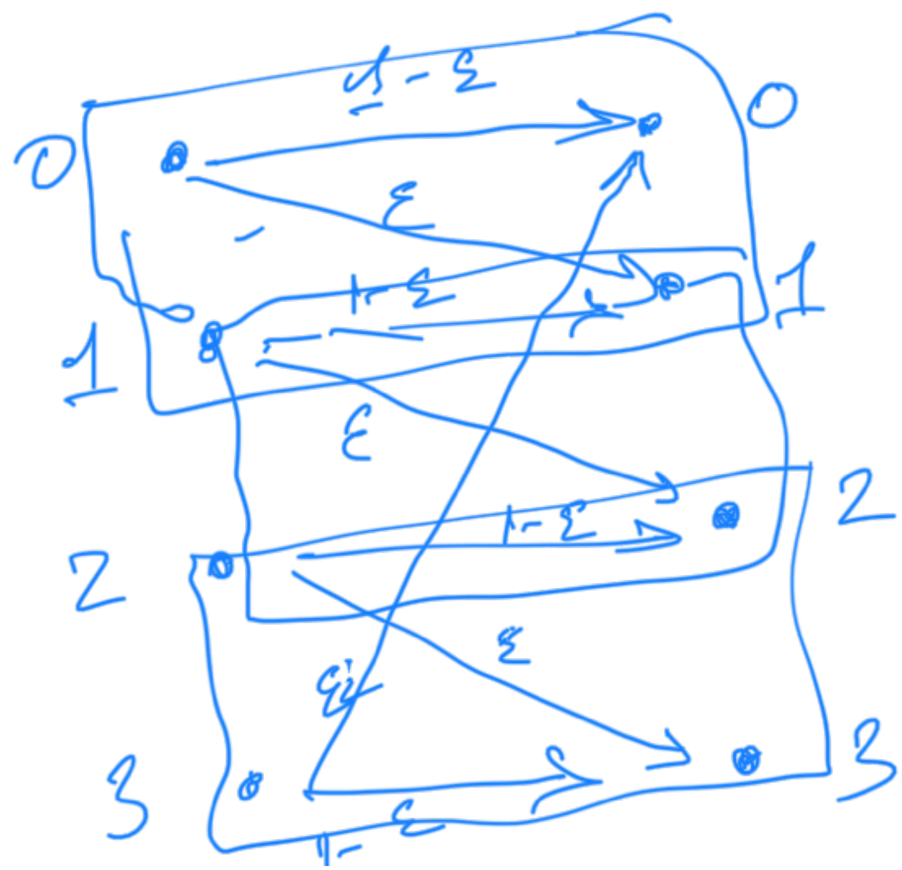
$$\boxed{C = 1 - H_b(\varepsilon)} \text{ bits.}$$

$$-\left[\frac{1}{2}\log\frac{1}{2}+\frac{1}{2}\log\frac{1}{2}\right]$$

$$= 1 \text{ bit}$$



Exp.



$$P(Y|X) = \begin{bmatrix} 1-\varepsilon & \varepsilon & 0 & 0 \\ 0 & 1-\varepsilon & \varepsilon & 0 \\ 0 & 0 & 1-\varepsilon & \varepsilon \\ \varepsilon & 0 & 0 & 1-\varepsilon \end{bmatrix}.$$

$$\begin{aligned} I(X;Y) &= H(Y) - H(Y|X) \\ &= H(Y) - \sum_x p(x) \underbrace{H(Y|X=x)}_{H(\varepsilon)} \\ &= \underbrace{H(Y)} - H(\varepsilon) \\ &\leq \log 4 - H(\varepsilon) \\ &= 2 - H(\varepsilon) \end{aligned}$$

If $p(x) = \frac{1}{4}$ for all 'x'.

$$p(y) = \frac{1}{4}\varepsilon + \frac{1}{4}(1-\varepsilon) = \frac{1}{4}$$

$$\boxed{C = 2 - H(\varepsilon)}$$??

Lec-14

Channel capacity.

BSC.

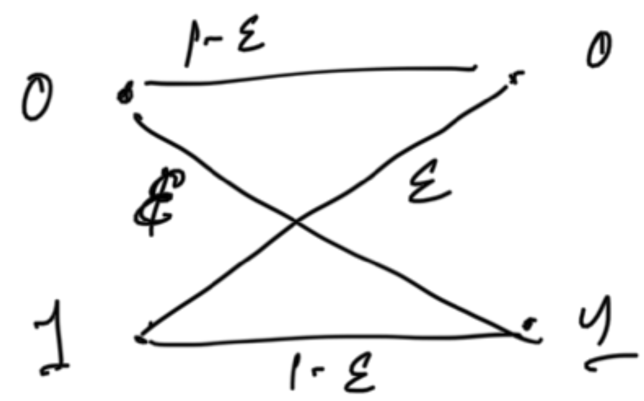
Channel matrix $= \begin{array}{cc} & y \\ x & \begin{bmatrix} p(y|x) \end{bmatrix} \end{array}$

Symmetric channel.

if any row of the channel matrix is

if any … other row.
a permutation of any other row
and the columns are also permutations
of each other.

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1-\varepsilon & \varepsilon \\ 1 & \varepsilon & 1-\varepsilon \end{array}$$



Doubly stochastic matrix :— is a non-negative matrix in which row sum and column sum are 1.

Weakly symmetric channel:

if any row is a permutation of any other row and sums of entries in each column are same/equal.

H.W. $C = \log|\mathcal{Y}| - H(\text{pmf corresponding to any row})$

Properties. (1) $C \geq 0$.

Pf. $C = \max_{p(x)} \underbrace{I(X;Y)}$

(2) $C = \max_{p(x)} I(X;Y)$

$\underbrace{I(X;Y)} = H(X) - \underbrace{H(X|Y)}$

$= H(Y) - H(Y|X)$

$\Rightarrow I(X;Y) \leq \max H(X) \leq \log|X|$

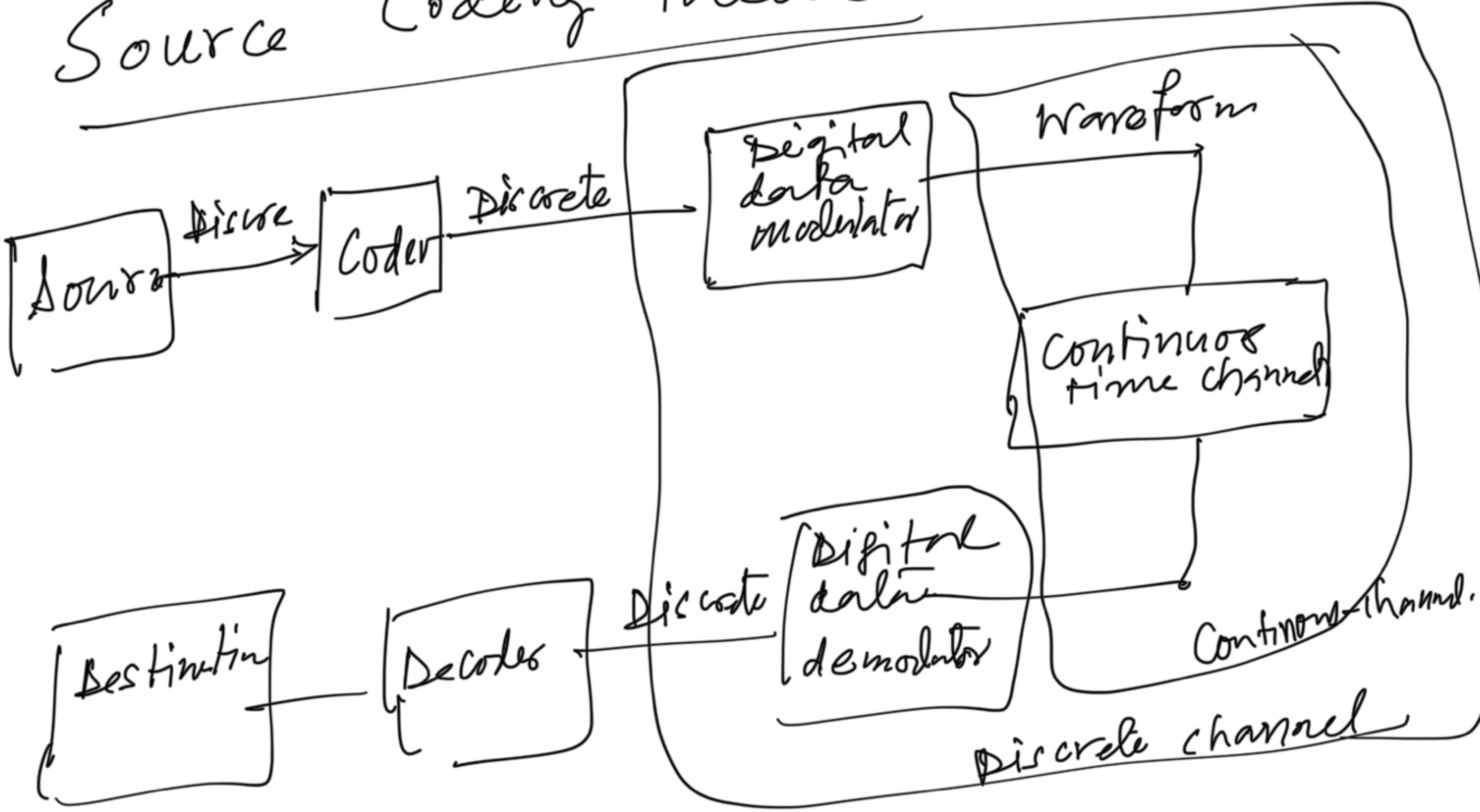
$\min I(X;Y) \leq H(Y) \leq \log|Y|$

$? \quad H(X) \leq \log|X|$

$$H(Y) \le \log|Y|$$

$$C \le \min\{\log|X|, \log|Y|\}$$

Source Coding theorem.



Discrete memoryless channel (DMC)

The inputs and outputs are sequences of letters and the output letter at a given time depends stochastically only on the corresponding input letter.

Source letters — $0, 1, \ldots, K-1$

output letters — $0, 1, \ldots, J-1$

'k' 'j' $p(j|k)$.

$\bar{x} \equiv (x_1, x_2 \ldots x_N)$

$\bar{y} \equiv (y_1 y_2 \cdots y_N)$

Q

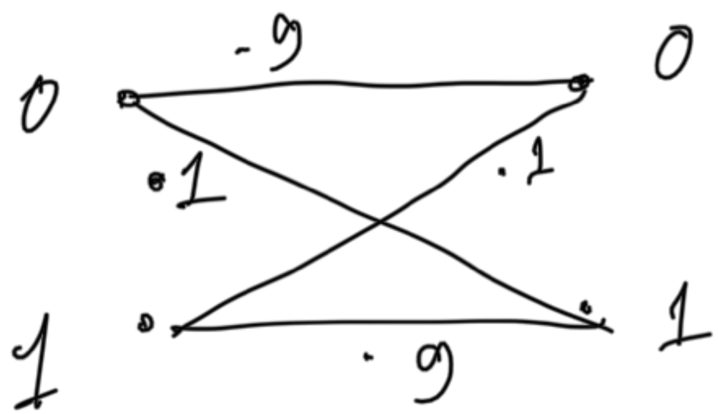
$\underline{x_k} \in \{0, \ldots, K-1\}$

$\underline{y_j} \in \{0, \ldots, J-1\}$.

$$\boxed{p(y_n | x_n)}$$

$$\boxed{P(\bar{y} | \bar{x})} = P((y_1 y_2 \cdots y_N) | (x_1 x_2 \cdots x_N)) = \boxed{\prod_{n=1}^{N} p(y_n | x_n)}$$

, Since the channel is <u>memoryless</u>.



$= (0.9) \times (0.9)$

$$P(\underset{\sim}{00}|\underset{\sim}{00}) = 0.81 = (0.9) \times (0.9)$$

$$P(\underset{\sim}{10}|\underset{\sim}{00}) = (0.1) \times (0.9) = 0.09$$

$$\bar{X}^2 = (X_1, X_2)$$

$$\bar{x} = \underbrace{(x_1 \ldots x_N)}$$

$$\bar{X}^{N} \equiv (X_1, X_2, \ldots, X_N)$$

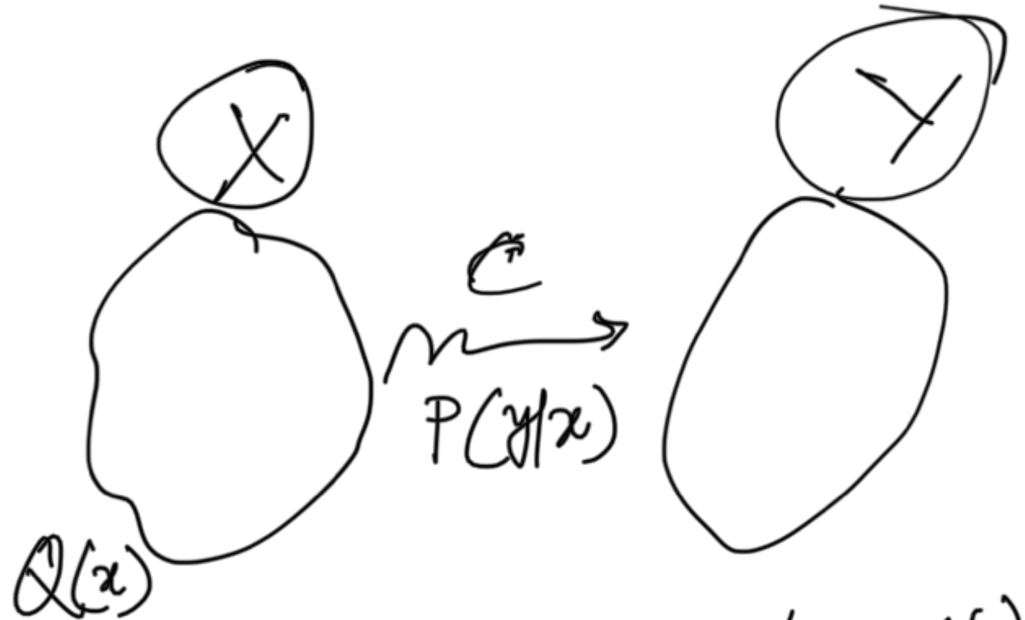
$$\bar{Y}^N \equiv (Y_1, Y_2, \ldots, Y_N), \quad \bar{y} \equiv (y_1, \ldots y_N)$$

Result.

$\bar{X}^N$, $\bar{Y}^N$

ⓐ $$I(\bar{X}^N; \bar{Y}^N) \le \underbrace{\sum_{n=1}^{N} I(\bar{X}_n; \bar{Y}_n)}\;??$$

ⓑ $$I(\bar{X}^N; \bar{Y}^N) \le N\boxed{C}$$



Pf.

$X_i \sim X$

$Y_j \sim Y$

$$I(\bar{X}^N ; \bar{Y}^N) = \underbrace{H(\bar{Y}^N)} - \underbrace{H(\bar{Y}^N | \bar{X}^N)}$$

$$H(\underbrace{\bar{Y}^N}_{Y} | \underbrace{\bar{X}^N}_{X}) = \sum_{\bar{x}, \bar{y}} Q(\bar{x}) P(\bar{y}|\bar{x}) \log \frac{1}{\underbrace{P(\bar{y}|\bar{x})}}$$

$$= \sum_{\bar{x}, \bar{y}} Q(\bar{x}) P(\bar{y}|\bar{x}) \sum_{n=1}^{N} \log \frac{1}{P(y_n | x_n)}$$

using the memoryless property

of the channel.

H.W.

$$= \sum_{n=1}^{N} H(Y_n | X_n) \quad \text{--- (1)}$$

$$H(\bar{Y}^N) = H(Y_1, Y_2, \ldots, Y_N)$$

$$\leq \sum_{n=1}^{N} H(Y_n) \quad \text{--- (2)}$$

$$I(\bar{X}^N; \bar{Y}^N) \leq \sum_{n=1}^{N} \left[H(Y_n) - H(Y_n | X_n) \right]$$

$$= \sum_{n=1}^{N} I(Y_n; X_n).$$

"$p(b_j|a_k)$"

$p(a_k)$

a_1 — b_1

a_2 — b_2

a_3 — b_3

$p(b_j) = ?$