Information and Coding Theory

MA41024/ MA60020/ MA60262

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Lecture 2
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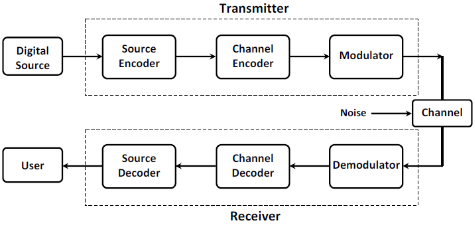
Content of the course

Model of a digital communication system



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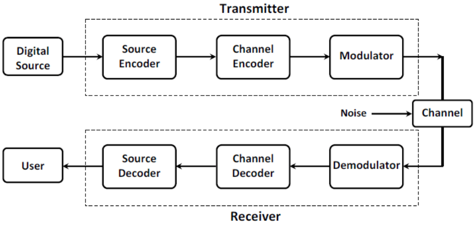

The encoding paradigm: Here

Entropy

Random variables Let Ω be a finite set, and $\mu : \Omega \to [0, 1]$ a function with

$$\sum_{\omega \in \Omega} \mu(\omega) = 1.$$

Then we refer Ω as the sample space and μ as a probability distribution (probability measure) on Ω.

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A random variable over Ω is any function $X : \Omega \to \mathbb{R}$. Then the expectation of X is

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If $\mathcal{X}$ is the range set of X then we can think of the probability distribution on $\mathcal{X}$ as

$$p(x) = \mathbb{P}[X = x] = \sum_{\omega : X(\omega) = x} \mu(\omega)$$

for any $x \in \mathcal{X}$

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Convex set and convex function

A set $S \subset \mathbb{R}^n$ is called convex if for any $\alpha \in [0, 1]$

$$\alpha \cdot x + (1 - \alpha) \cdot y \in S$$

for any $x, y \in S$.

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Let S be a convex subset of $\mathbb{R}^n$. Then a function $f : S \to \mathbb{R}$ is said to be a convex function on S if for all $\alpha \in [0, 1]$

$$f(\alpha \cdot x + (1 - \alpha) \cdot y) \leq \alpha \cdot f(x) + (1 - \alpha) \cdot f(y)$$

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is a convex subset of $\mathbb{R}^{n+1}$. If the inequality is strict then f is called strictly convex.

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Note A function $f : \mathbb{R} \to \mathbb{R}$ is a twice differentiable function then f is convex if and only if $f"(x) \geq 0$ for all $x \in S$

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Homework $f(x) = x^2$ is a convex function on $\mathbb{R}$.
$f(x) = \log(x), f(x) = x\log(x)$ are concave and convex functions respectively, on $(0, \infty)$

Entropy

Observation Consider a rv X which takes the values x with probability α and y with probability $1-\alpha$. Let $f(X)$ be a function. Then what happens to the convexity condition?

Jensen's inequality Let $S \subseteq \mathbb{R}^n$ be a convex set and let X be a random variable whose range set is a subset of S. Then for a convex function $f : S \to \mathbb{R}$,

$$\mathbb{E}[f(X)] \geq f(\mathbb{E}[X]).$$

Equivalently, for a concave function $f : S \to \mathbb{R}$,

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Homework Prove Cauchy-Schwarz inequality using Jensen's inequality.

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Shannon's assumptions - Let $\mathcal{X} = \{a_1, \ldots, a_n\}$ with pmf $p(a_i), 1 \leq i \leq n$. Denote the entropy by H

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Question How much information is revealed when we know outcome of a random experiment?

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Question How much information is revealed when we know outcome of a random experiment? How surprised are we?

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Entropy Suppose X is a rv distributed over $\mathcal{X} = \{a_1, \ldots, a_n\}$ such that each value $x \in \mathcal{X}$ occurs with probability $p(x)$. Then the entropy of X is

$$H(X) = \sum_{x \in \mathcal{X}} p(x) \cdot \underbrace{\log_2 \left(\frac{1}{p(x)} \right)}_{\text{surprise}}$$

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