Information and Coding Theory

MA41024/ MA60020/ MA60262

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Lecture 10
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Fano's inequality

For a Markov chain $X \to Y \to \widehat{X}$, interpret X as the choice of an unknown parameter from some finite set $\mathcal{X}$, Y as the data generated from this, say sequence of independent samples, and $\widehat{X}$ is a guess for X, which depends only on the data.

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Fano's inequality Let $X \to Y \to \widehat{X}$ be a Markov chain, and let $p_e = \mathbb{P}[\widehat{X} \neq X]$. Suppose $H(p_e)$ denotes the entropy function computed at p_e. Then

$$H(p_e) + p_e \cdot \log(|\mathcal{X}| - 1) \geq H(X|\widehat{X}) \geq H(X|Y)$$

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Proof of Fano's inequality Define a rv E corresponding to the error i.e. $E = 1$ if $\widehat{X} = X$ and $E = 0$ if $\widehat{X} = X$.

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since $H(E|X, \widehat{X}) = 0$.

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&\leq H(p_e) + p_e \cdot \log(|\mathcal{X}| - 1)
\end{aligned}$$