Indian Institute of Technology Kharagpur
Course: MA41024/MA60020 Information and Coding Theory
Spring Semester 2020-21
Time : 45 minutes
Class Test - I

Declaration:

- **Each question carries 2 marks.**
- **NO query will be entertained during the examination.**
- **There may be multiple options correct for a problem. Full marks is given only when all the correct options are identified.**
- **Once a problem is passed, it will not appear in your screen again and hence if a problem appears in your screen then identify the correct option and then go for the next problem.**

---

1. Let an information source X produce a symbol x from the alphabet $\mathcal{X} = \{0, 1, \ldots, 9, a, b, \ldots, y, z\}$ with probability
$$p(x) = \begin{cases} \frac{1}{3} \text{ if } x \in A = \{0, \ldots, 9\} \\ \frac{1}{3} \text{ if } x \in B = \{a, e, i, o, u\} \\ \frac{1}{3}, \text{ otherwise} \end{cases}$$
Assume that all the symbols in A, B and otherwise are equiprobable. Then the entropy of X is $-------$ bits (up to two decimal digits)

Ans. $\log 3 + \frac{1}{3}(\log 10 + \log 5 + \log 21) = \log 3 + \frac{1}{3} \log 1050 = 1.5849625007212 + \frac{1}{3} \times 10.036173612553 = 1.5849625007212 + 3.34539120418 = 4.93035370491 \in [4.90, 4.95]$.

2. Let X be a discrete random variable whose entropy $H(X) = 4$ bits. Then consider a new random variable $Y = X + 2$. Then

   (a) $H(X|Y) = ----$ and $H(XY) = ---$

   Ans. 0 and $H(XY) = H(X) + H(Y|X) = 4$

3. Consider a binary channel with source symbols a_1, a_2 with $p(a_1) = 0.3$. Assume that the output symbols are b_1, b_2. Let the noisy channel be defined by the conditional probabilities $p(b_1|a_1) = 0.4, p(b_2|a_1) = 0.6$, $p(b_1|a_2) = 0.75$, $p(b_2|a_2) = 0.25$. Then
$$I(X;Y) = ---$$
(up to four decimal digits) where X and Y denote the input and output random variables respectively.

   Ans. Obviously, $p(a_2) = 0.7$ and hence $p(b_1) = p(b_1|a_1)p(a_1) + p(b_1|a_2)p(a_2) = (0.4 \times 0.3) + (0.75 \times 0.7) = 0.645$. Similarly, $p(b_2) = .355$ The joint distribution is as follows:
$$p(a_1, b_1) = p(a_1)p(b_1|a_1) = 0.3 \times 0.4 = 0.12, p(a_1, b_2) = p(a_1)p(b_2|a_1) = 0.3 \times 0.6 = 0.18$$
$$p(a_2, b_1) = p(a_2) \times p(b_1|a_2) = 0.7 \times 0.75 = .525, p(a_2, b_2) = p(a_2) \times p(b_2|a_2) = 0.7 \times 0.25 = 0.175.$$

Then

$$I(X;Y) = \sum_{i=1}^{2}\sum_{j=1}^{2} p(a_i, b_j) \log \frac{p(b_j|a_i)}{p(b_j)} = 0.0793 \in [.0790, 0.0795] \text{ bits}$$

4. Let X and Y be two independent random variables. Then which of the following is true?

   (a) $H(3X, -3Y) = 3H(X) - 3H(Y)$
   (b) $H(3X, -3Y) = H(X) + H(Y)$
   (c) $H(3X, -3Y) = H(X) - H(Y)$
   (d) None of the above

   Ans. b)

5. Let X be the random variable which represents an information source that produces letters from an alphabet $\{0, 1, 2\}$, with $p(0) = 1/4$, $p(1) = 1/4$ and $p(2) = 1/2$. Let each letter be transmitted through two separate channels C_1 and C_2 simultaneously with outputs y and z respectively, where $y, z \in \{0, 1\}$. Let us denote the output random variables corresponding to C_1 and C_2 as Y and Z, and the channels are defined by the following transition probabilities:

$$\begin{cases} C_1 : p(0|0) = 1, p(1|1) = 1, p(0|2) = 1/2, p(1|2) = 1/2 \\ C_2 : p(0|0) = 1, p(0|1) = 1, p(1|2) = 1 \end{cases}$$

   Then

   (a) $H(Y|Z) = - - -$
   (b) $I(X;Y) = - - -$
   (c) $I(X;Z) = - - -$
   (d) $I(X;Y|Z) = - - - - -$
   (e) $I(X;Y,Z) = - - - - -$

   Ans. $P_Y(0) = 1/2 = P_Y(1)$ and $P_Z(0) = 1/2 = P_Z(1)$. If $Z = 1$, then $X = 2$, and $Y = 0, 1$ with equal probability; if $Z = 0$, then $X = 0, 1$ with equal probability and as a consequence, $Y = 0, 1$ with equal probability. Therefore,

$$H(Y|Z) = P_Z(0)H(Y|Z = 0) + P_Z(1)H(Y|Z = 1) = 1.$$

   Since $H(Y|X = 0) = H(Y|X = 1) = 0$ and $H(Y|X = 2) = 1$, we have

$$I(X;Y) = H(Y) - H(Y|X) = 1 - 1/2 = 0.5$$

   Since $H(Z|X) = 0$, we have

$$I(X;Z) = H(Z) - H(Z|X) = H(Z) = 1.$$

   Since Z is completely determined by X, $H(Y|X,Z) = H(Y|X) = 0.5$. We have

$$I(X;Y|Z) = H(Y|Z) - H(Y|X,Z) = 1 - 0.5 = 0.5$$

   and

$$I(X;Y,Z) = I(X;Z) + I(X;Y|Z) = 1 + 0.5 = 1.5$$

6. Consider the Binary Symmetric Channel with information source symbols $0, 1$ with $p(0) \neq 1/2$ and the error probability of the channel $p_e \neq 1/2$. Let X and Y denote the input and output random variables respectively. Then which of the following are true?

   (a) $H(X|Y) = H(Y|X)$

   (b) $H(X|Y) \leq H(Y|X)$

   (c) $H(Y|X) \leq H(X|Y)$

   (d) None of the above

   Ans. b) Note that $H(X) \leq H(Y)$

7. Let the random variable X represent an information source with alphabet $\{0, 1\}$ and $p(1) = 0.9$. Consider the sequence $\overline{u}_n = (1, 1, \ldots, 1)$ of length n which is large. Then which of the following is true?

   (a) $\overline{u}_n$ is a typical sequence

   (b) $\overline{u}_n$ is not a typical sequence.

   Ans. b) since $\frac{1}{n} \log p(1, ..., 1) = \log 1/0.9 \approx 0.11$ which is not close to $H(X) = 0.46$ when $n \to \infty$. This contradicts the AEP.