Big Data Analysis
(MA60306)

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Lecture 7
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Computing with data

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Question Is $\|M\|_F = \sqrt{\sum_{i,j} |m_{ij}|^2}$ a norm, where $M = [m_{ij}]$?

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Now note that $\kappa\left(\begin{bmatrix} 4.1 & 2.8 \\ 9.7 & 6.6 \end{bmatrix}\right) = 1.6230e + 03$ (too big!! for which choice of the norm?)

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Question What is the condition number of an orthogonal matrix?

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It can also be shown that

$$\frac{\|(A+\triangle A)^{-1}-A^{-1}\|}{\|A^{-1}\|} \leq \kappa(A)\frac{\|\triangle A\|}{\|A\|}$$

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Question What is the conclusion?

```
import numpy as np
A = np.array([[1, 2.0000000001],[2, 4]])
b=np.array([1, 2])
b2=np.array([1, 2.01])
np.linalg.cond(A)
x = np.linalg.solve(A, b) print(x)
x2 = np.linalg.solve(A, b2) print(x2)
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Example Let $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Then

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Question Is the converse true?

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Theorem (Cholesky decomposition theorem) Any positive definite matrix A can be decomposed in exactly one way as $A = R^T R$ for some upper triangular matrix R whose diagonal entries are positive. R is called the Cholesky factor of A.

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Question How do we obtain R?

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} r_{11} & 0 & 0 & \dots & 0 \\ r_{12} & r_{22} & 0 & \dots & 0 \\ r_{13} & r_{23} & r_{33} & & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ r_{1n} & r_{2n} & r_{3n} & \dots & r_{nn} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & \dots & r_{1n} \\ 0 & r_{22} & r_{23} & \dots & r_{2n} \\ 0 & 0 & r_{33} & \dots & r_{3n} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & r_{nn} \end{bmatrix}$$

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$\rightarrow$ Thus, $r_{2j} = (a_{2j} - r_{12}r_{1j})/r_{22}$, $j = 3, \ldots, n$

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The recipe for calculating R - the Cholesky's method

$$
\begin{aligned}
r_{ii} &= +\sqrt{a_{ii} - \sum_{k=1}^{i-1} r_{ki}^2} \\
r_{ij} &= \left(a_{ij} - \sum_{k=1}^{i-1} r_{ki} r_{kj}\right) / \; r_{ii}, j = i+1, \ldots, n
\end{aligned}
$$

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r_{ij} &= \left(a_{ij} - \sum_{k=1}^{i-1} r_{ki} r_{kj} \right) / \ r_{ii}, j = i+1, \ldots, n
\end{aligned}
$$

Question Is it a backward stable algorithm?

Cholesky decomposition

The recipe for calculating R - the Cholesky's method

$$
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Question Is it a backward stable algorithm?
See Higham's book for a proof

Cholesky's algorithm

Flop count - The upper part of R will be stored over the upper part of A

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for $i = 1, \ldots, n$
  for $k = 1, \ldots, i-1$
    $a_{ii} \leftarrow a_{ii} - a_{ki}^2$
  if $a_{ii} \leq 0$, set error flag
  $a_{ii} \leftarrow \sqrt{a_{ii}}$ (this is r_{ii})
  for $j = i+1, \ldots, n$
    for $k = 1, \ldots, i-1$ (not executed when $i = 1$)
      $a_{ij} \leftarrow a_{ij} - a_{ki}a_{kj}$
    $a_{ij} \leftarrow a_{ij}/a_{ii}$ (this is r_{ij})

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→ # of flops in the second k loop: $\sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=1}^{i-1} 2$

Cholesky's algorithm

$$
\begin{aligned}
\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=1}^{i-1} 2 &= 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} (i-1) \\
&= 2 \sum_{i=1}^{n} (n-i)(i-1) \\
&= 2n \sum_{i=1}^{n} (i-1) - 2 \sum_{i=1}^{n} i^2 + 2 \sum_{i=1}^{n} i \\
&= n^3 - 2\frac{n^3}{3} + O(n^2) \\
&\approx \frac{n^3}{3}
\end{aligned}
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Cholesky's algorithm

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Question How many flops are needed to compute the forward and backward substitution?