Big Data Analysis
(MA60306)

Bibhas Adhikari

Spring 2022-23, IIT Kharagpur

Lecture 5
January 12, 2023

Regression models

Theorem If $\text{Loss}(y, \widehat{y}) = (y - \widehat{y})^2$ then the optimal prediction function g^* is equal to the conditional expectation of Y given $\mathbf{X} = \mathbf{x}$:

$$g^*(\mathbf{x}) = \mathbb{E}[Y|\mathbf{X} = \mathbf{x}]$$

Regression models

Theorem If $\text{Loss}(y, \widehat{y}) = (y - \widehat{y})^2$ then the optimal prediction function g^* is equal to the conditional expectation of Y given $\mathbf{X} = \mathbf{x}$:

$$g^*(\mathbf{x}) = \mathbb{E}[Y|\mathbf{X} = \mathbf{x}]$$

Consequence

- The conditional $\mathbf{X} = \mathbf{x}$, the random response Y can be written as

$$Y = g^*(\mathbf{x}) + \epsilon(\mathbf{x})$$

where $\epsilon(\mathbf{x})$ can be thought of as a random deviation of the response from its conditional mean at $\mathbf{x}$.

Regression models

Theorem If $\text{Loss}(y, \widehat{y}) = (y - \widehat{y})^2$ then the optimal prediction function g^* is equal to the conditional expectation of Y given $\mathbf{X} = \mathbf{x}$:

$$g^*(\mathbf{x}) = \mathbb{E}[Y|\mathbf{X} = \mathbf{x}]$$

Consequence

- The conditional $\mathbf{X} = \mathbf{x}$, the random response Y can be written as

$$Y = g^*(\mathbf{x}) + \epsilon(\mathbf{x})$$

  where $\epsilon(\mathbf{x})$ can be thought of as a random deviation of the response from its conditional mean at $\mathbf{x}$.
- $\mathbb{E}[\epsilon(\mathbf{x})] = 0$

Regression models

Theorem If $\text{Loss}(y, \widehat{y}) = (y - \widehat{y})^2$ then the optimal prediction function g^* is equal to the conditional expectation of Y given $\mathbf{X} = \mathbf{x}$:

$$g^*(\mathbf{x}) = \mathbb{E}[Y|\mathbf{X} = \mathbf{x}]$$

Consequence

- The conditional $\mathbf{X} = \mathbf{x}$, the random response Y can be written as

$$Y = g^*(\mathbf{x}) + \epsilon(\mathbf{x})$$

  where $\epsilon(\mathbf{x})$ can be thought of as a random deviation of the response from its conditional mean at $\mathbf{x}$.
- $\mathbb{E}[\epsilon(\mathbf{x})] = 0$
- $\mathbb{V}\text{ar}[\epsilon(\mathbf{x})] = \nu^2(\mathbf{x})$ for some function $\nu(\mathbf{x})$

Regression models

Geometry Relate it to a data set: Where x measures height and y measures age of a person. Suppose we want to write y as a function of x through the predictor function:

$$y = g^*(x) + \epsilon(x),$$

Regression models

Geometry Relate it to a data set: Where x measures height and y measures age of a person. Suppose we want to write y as a function of x through the predictor function:

$$y = g^*(x) + \epsilon(x), \; := \underbrace{\beta_0 + \beta_1 x + \epsilon(x)}_{\text{regression line}}$$

Regression models

Geometry Relate it to a data set: Where x measures height and y measures age of a person. Suppose we want to write y as a function of x through the predictor function:

$$y = g^*(x) + \epsilon(x), \; := \underbrace{\beta_0 + \beta_1 x + \epsilon(x)}_{\text{regression line}}$$

It may so happen that $(x, y) \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\boldsymbol{\mu} = (\mu_x, \mu_y)$ and $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}$

Regression models

Geometry Relate it to a data set: Where x measures height and y measures age of a person. Suppose we want to write y as a function of x through the predictor function:

$$y = g^*(x) + \epsilon(x), \ := \underbrace{\beta_0 + \beta_1 x + \epsilon(x)}_{\text{regression line}}$$

It may so happen that $(x, y) \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\boldsymbol{\mu} = (\mu_x, \mu_y)$ and $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}$

Recall Bivariate normal distribution

$$f(x, y) = \frac{1}{2\pi\sqrt{(1-\rho^2)}\sigma_x\sigma_y} \times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right\}$$

where $\sigma_x > 0$, $\sigma_y > 0$, and $|\rho| < 1$.

Regression model

Note that if x denotes the total marks that one student has obtained and y denotes the grade then it becomes a classification problem.

Regression model

Note that if x denotes the total marks that one student has obtained and y denotes the grade then it becomes a classification problem.

Multivariate normal distribution Let $\mathbf{\Sigma}$ be a positive definite $n \times n$ matrix and $\boldsymbol{\mu}$ an n dimensional vector. Then the pdf of a multivariate normal rv $\mathbf{X} = (X_1, \ldots, X_n)$ i.e. $\mathbf{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \mathbf{\Sigma})$ is

$$f(x_1, \ldots, x_n) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{\Sigma})}} \exp\left[-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \mathbf{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right]$$

Regression model

Note that if x denotes the total marks that one student has obtained and y denotes the grade then it becomes a classification problem.

Multivariate normal distribution Let $\mathbf{\Sigma}$ be a positive definite $n \times n$ matrix and $\boldsymbol{\mu}$ an n dimensional vector. Then the pdf of a multivariate normal rv $\mathbf{X} = (X_1, \ldots, X_n)$ i.e. $\mathbf{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \mathbf{\Sigma})$ is

$$f(x_1, \ldots, x_n) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{\Sigma})}} \exp\left[-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \mathbf{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right]$$

$\rightarrow$ $\mathbb{E}[\mathbf{X}] = \boldsymbol{\mu}$, $\mathbb{V}\text{ar}[\mathbf{X}] = \mathbf{\Sigma} = [\Sigma_{ij}]$

Regression model

Note that if x denotes the total marks that one student has obtained and y denotes the grade then it becomes a classification problem.

Multivariate normal distribution Let $\mathbf{\Sigma}$ be a positive definite $n \times n$ matrix and $\boldsymbol{\mu}$ an n dimensional vector. Then the pdf of a multivariate normal rv $\mathbf{X} = (X_1, \ldots, X_n)$ i.e. $\mathbf{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \mathbf{\Sigma})$ is

$$f(x_1, \ldots, x_n) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{\Sigma})}} \exp\left[-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \mathbf{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right]$$

$\rightarrow$ $\mathbb{E}[\mathbf{X}] = \boldsymbol{\mu}$, $\mathbb{V}\text{ar}[\mathbf{X}] = \mathbf{\Sigma} = [\Sigma_{ij}]$ where $\Sigma_{ij} = \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)]$

Regression model

Note that if x denotes the total marks that one student has obtained and y denotes the grade then it becomes a classification problem.

Multivariate normal distribution Let $\boldsymbol{\Sigma}$ be a positive definite $n \times n$ matrix and $\boldsymbol{\mu}$ an n dimensional vector. Then the pdf of a multivariate normal rv $\mathbf{X} = (X_1, \ldots, X_n)$ i.e. $\mathbf{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is

$$f(x_1, \ldots, x_n) = \frac{1}{\sqrt{(2\pi)^n \det(\boldsymbol{\Sigma})}} \exp\left[-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right]$$

- $\rightarrow$ $\mathbb{E}[\mathbf{X}] = \boldsymbol{\mu}$, $\mathbb{V}\mathrm{ar}[\mathbf{X}] = \boldsymbol{\Sigma} = [\Sigma_{ij}]$ where $\Sigma_{ij} = \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)]$
- $\rightarrow$ Let $X_1, \ldots, X_n$ be independent $\mathcal{N}(0, 1)$ random variables. Then $\mathbf{X} = (X_1, \ldots, X_n)$ has the density function $\mathcal{N}_n(\mathbf{0}, \mathbf{I}_n)$

Regression model

Note that if x denotes the total marks that one student has obtained and y denotes the grade then it becomes a classification problem.

Multivariate normal distribution Let $\mathbf{\Sigma}$ be a positive definite $n \times n$ matrix and $\boldsymbol{\mu}$ an n dimensional vector. Then the pdf of a multivariate normal rv $\mathbf{X} = (X_1, \ldots, X_n)$ i.e. $\mathbf{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \mathbf{\Sigma})$ is

$$f(x_1, \ldots, x_n) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{\Sigma})}} \exp\left[-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \mathbf{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right]$$

- $\rightarrow$ $\mathbb{E}[\mathbf{X}] = \boldsymbol{\mu}$, $\mathbb{V}\mathrm{ar}[\mathbf{X}] = \mathbf{\Sigma} = [\Sigma_{ij}]$ where $\Sigma_{ij} = \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)]$
- $\rightarrow$ Let $X_1, \ldots, X_n$ be independent $\mathcal{N}(0, 1)$ random variables. Then $\mathbf{X} = (X_1, \ldots, X_n)$ has the density function $\mathcal{N}_n(\mathbf{0}, \mathbf{I}_n)$
- $\rightarrow$ Let $\mathbf{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \mathbf{\Sigma})$, C be an $m \times n$ matrix of rank m, and d be an m dimensional vector. Then $C\mathbf{X} + d \sim \mathcal{N}_m(C\boldsymbol{\mu} + d, C\mathbf{\Sigma}C^T)$

Regression model

Note that if x denotes the total marks that one student has obtained and y denotes the grade then it becomes a classification problem.

Multivariate normal distribution Let $\mathbf{\Sigma}$ be a positive definite $n \times n$ matrix and $\boldsymbol{\mu}$ an n dimensional vector. Then the pdf of a multivariate normal rv $\mathbf{X} = (X_1, \ldots, X_n)$ i.e. $\mathbf{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \mathbf{\Sigma})$ is

$$f(x_1, \ldots, x_n) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{\Sigma})}} \exp\left[-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \mathbf{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right]$$

- $\rightarrow$ $\mathbb{E}[\mathbf{X}] = \boldsymbol{\mu}$, $\mathbb{V}\mathrm{ar}[\mathbf{X}] = \mathbf{\Sigma} = [\Sigma_{ij}]$ where $\Sigma_{ij} = \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)]$
- $\rightarrow$ Let $X_1, \ldots, X_n$ be independent $\mathcal{N}(0, 1)$ random variables. Then $\mathbf{X} = (X_1, \ldots, X_n)$ has the density function $\mathcal{N}_n(\mathbf{0}, \mathbf{I}_n)$
- $\rightarrow$ Let $\mathbf{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \mathbf{\Sigma})$, C be an $m \times n$ matrix of rank m, and d be an m dimensional vector. Then $C\mathbf{X} + d \sim \mathcal{N}_m(C\boldsymbol{\mu} + d, C\mathbf{\Sigma}C^T)$
- $\rightarrow$ If $\mathbf{X} = A\mathbf{Z} + \boldsymbol{\mu}$ where A is an $n \times n$ nonsingular matrix and $\mathbf{Z} \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I}_n)$ then $\mathbf{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, AA^T)$

Regression models

Method of least square Recall $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}(\mathbf{x})$

$$\min_{\boldsymbol{\beta}} \sum_i \epsilon_i^2$$

Regression models

Method of least square Recall $\mathbf{Y} = \mathbf{X}\beta + \boldsymbol{\epsilon}(\mathbf{x})$

$$\min_{\boldsymbol{\beta}} \sum_i \epsilon_i^2$$

$$i.e. \min_{\boldsymbol{\theta} \in \mathcal{C}(\mathbf{X}) = \Omega} \boldsymbol{\epsilon}^T \boldsymbol{\epsilon} = \|\mathbf{Y} - \boldsymbol{\theta}\|^2$$

where $\boldsymbol{\theta} = \mathbf{X}\beta$ and Ω is the column space of $\mathbf{X}$ i.e. $\Omega = \{\mathbf{y} : \mathbf{y} = \mathbf{X}\mathbf{x} \text{ for any } \mathbf{x}\}$

Regression models

Method of least square Recall $\mathbf{Y} = \mathbf{X}\beta + \epsilon(\mathbf{x})$

$$\min_{\beta} \sum_i \epsilon_i^2$$

$$i.e. \min_{\theta \in \mathcal{C}(\mathbf{X}) = \Omega} \epsilon^T \epsilon = \|\mathbf{Y} - \theta\|^2$$

where $\theta = \mathbf{X}\beta$ and Ω is the column space of $\mathbf{X}$ i.e.
$\Omega = \{\mathbf{y} : \mathbf{y} = \mathbf{X}\mathbf{x} \text{ for any } \mathbf{x}\}$

From the geometry, what is your guess for θ which can minimize the function?

Regression models

Method of least square Recall $\mathbf{Y} = \mathbf{X}\beta + \boldsymbol{\epsilon}(\mathbf{x})$

$$\min_{\boldsymbol{\beta}} \sum_i \epsilon_i^2$$

$$i.e. \min_{\boldsymbol{\theta} \in \mathcal{C}(\mathbf{X}) = \Omega} \boldsymbol{\epsilon}^T \boldsymbol{\epsilon} = \|\mathbf{Y} - \boldsymbol{\theta}\|^2$$

where $\boldsymbol{\theta} = \mathbf{X}\beta$ and Ω is the column space of $\mathbf{X}$ i.e.
$\Omega = \{\mathbf{y} : \mathbf{y} = \mathbf{X}\mathbf{x} \text{ for any } \mathbf{x}\}$

From the geometry, what is your guess for $\boldsymbol{\theta}$ which can minimize the function?

Obviously, $\widehat{\boldsymbol{\theta}} = \boldsymbol{\theta}$ will minimize $\|\mathbf{Y} - \boldsymbol{\theta}\|^2$ if $(\mathbf{Y} - \widehat{\boldsymbol{\theta}}) \perp \Omega$

Regression models

Obviously. $\widehat{\boldsymbol{\theta}}$ can be obtained via a projection matrix P, namely $\widehat{\boldsymbol{\theta}} = P\mathbf{Y}$, where P is the orthogonal projection onto Ω i.e. $P\boldsymbol{\theta} = \boldsymbol{\theta}$. $P^T = P$ and $P^2 = P$

Regression models

Obviously. $\widehat{\boldsymbol{\theta}}$ can be obtained via a projection matrix P, namely $\widehat{\boldsymbol{\theta}} = P\mathbf{Y}$, where P is the orthogonal projection onto Ω i.e. $P\boldsymbol{\theta} = \boldsymbol{\theta}$. $P^T = P$ and $P^2 = P$

Then

$$\mathbf{Y} - \boldsymbol{\theta} = (\mathbf{Y} - \widehat{\boldsymbol{\theta}}) + (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$$

and

$$\begin{aligned}
(\mathbf{Y} - \widehat{\boldsymbol{\theta}})^T(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) &= (\mathbf{Y} - P\mathbf{Y})^T P(\mathbf{Y} - \boldsymbol{\theta}) \\
&= \mathbf{Y}^T(I_n - P)P(\mathbf{Y} - \boldsymbol{\theta}) \\
&= 0.
\end{aligned}$$

Regression models

Obviously. $\widehat{\boldsymbol{\theta}}$ can be obtained via a projection matrix P, namely $\widehat{\boldsymbol{\theta}} = P\mathbf{Y}$, where P is the orthogonal projection onto Ω i.e. $P\boldsymbol{\theta} = \boldsymbol{\theta}$. $P^T = P$ and $P^2 = P$

Then

$$\mathbf{Y} - \boldsymbol{\theta} = (\mathbf{Y} - \widehat{\boldsymbol{\theta}}) + (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$$

and

$$\begin{aligned} (\mathbf{Y} - \widehat{\boldsymbol{\theta}})^T(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) &= (\mathbf{Y} - P\mathbf{Y})^T P(\mathbf{Y} - \boldsymbol{\theta}) \\ &= \mathbf{Y}^T(I_n - P)P(\mathbf{Y} - \boldsymbol{\theta}) \\ &= 0. \end{aligned}$$

Thus

$$\|\mathbf{Y} - \boldsymbol{\theta}\|^2 = \|Y - \widehat{\boldsymbol{\theta}}\|^2 + \|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\|^2 \geq \|\mathbf{Y} - \widehat{\boldsymbol{\theta}}\|^2,$$

with equality iff $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}$.

Regression models

Now since $\mathbf{Y} - \widehat{\boldsymbol{\theta}}$ is perpendicular to Ω,

$$\mathbf{X}^T(\mathbf{Y} - \widehat{\boldsymbol{\theta}}) = 0 \text{ i.e. } \mathbf{X}^T\widehat{\boldsymbol{\theta}} = \mathbf{X}^T\mathbf{Y}$$

Regression models

Now since $\mathbf{Y} - \widehat{\boldsymbol{\theta}}$ is perpendicular to Ω,

$$\mathbf{X}^T(\mathbf{Y} - \widehat{\boldsymbol{\theta}}) = 0 \text{ i.e. } \mathbf{X}^T\widehat{\boldsymbol{\theta}} = \mathbf{X}^T\mathbf{Y}$$

Note that $\widehat{\boldsymbol{\theta}}$ is uniquely determined!!

Regression models

Now since $\mathbf{Y} - \widehat{\boldsymbol{\theta}}$ is perpendicular to Ω,

$$\mathbf{X}^T(\mathbf{Y} - \widehat{\boldsymbol{\theta}}) = 0 \text{ i.e. } \mathbf{X}^T\widehat{\boldsymbol{\theta}} = \mathbf{X}^T\mathbf{Y}$$

Note that $\widehat{\boldsymbol{\theta}}$ is uniquely determined!!

If columns of $\mathbf{X}$ are linearly independent then there exists a unique $\widehat{\boldsymbol{\beta}}$ such that $\widehat{\boldsymbol{\theta}} = \mathbf{X}\widehat{\boldsymbol{\beta}}$.

Regression models

Now since $\mathbf{Y} - \widehat{\boldsymbol{\theta}}$ is perpendicular to Ω,

$$\mathbf{X}^T(\mathbf{Y} - \widehat{\boldsymbol{\theta}}) = 0 \text{ i.e. } \mathbf{X}^T\widehat{\boldsymbol{\theta}} = \mathbf{X}^T\mathbf{Y}$$

Note that $\widehat{\boldsymbol{\theta}}$ is uniquely determined!!

If columns of $\mathbf{X}$ are linearly independent then there exists a unique $\widehat{\boldsymbol{\beta}}$ such that $\widehat{\boldsymbol{\theta}} = \mathbf{X}\widehat{\boldsymbol{\beta}}$.

Hence

$$\mathbf{X}^T\mathbf{X}\widehat{\boldsymbol{\beta}} = \mathbf{X}^T\mathbf{y},$$

which is known as the normal equation.

Regression models

Now since $\mathbf{Y} - \widehat{\boldsymbol{\theta}}$ is perpendicular to Ω,

$$\mathbf{X}^T(\mathbf{Y} - \widehat{\boldsymbol{\theta}}) = 0 \text{ i.e. } \mathbf{X}^T\widehat{\boldsymbol{\theta}} = \mathbf{X}^T\mathbf{Y}$$

Note that $\widehat{\boldsymbol{\theta}}$ is uniquely determined!!

If columns of $\mathbf{X}$ are linearly independent then there exists a unique $\widehat{\boldsymbol{\beta}}$ such that $\widehat{\boldsymbol{\theta}} = \mathbf{X}\widehat{\beta}$.

Hence

$$\mathbf{X}^T\mathbf{X}\widehat{\beta} = \mathbf{X}^T\mathbf{y},$$

which is known as the normal equation. Thus (why)

$$\widehat{\beta} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}$$

Regression models

Now since $\mathbf{Y} - \widehat{\boldsymbol{\theta}}$ is perpendicular to Ω,

$$\mathbf{X}^T(\mathbf{Y} - \widehat{\boldsymbol{\theta}}) = 0 \text{ i.e. } \mathbf{X}^T\widehat{\boldsymbol{\theta}} = \mathbf{X}^T\mathbf{Y}$$

Note that $\widehat{\boldsymbol{\theta}}$ is uniquely determined!!

If columns of $\mathbf{X}$ are linearly independent then there exists a unique $\widehat{\boldsymbol{\beta}}$ such that $\widehat{\boldsymbol{\theta}} = \mathbf{X}\widehat{\boldsymbol{\beta}}$.

Hence

$$\mathbf{X}^T\mathbf{X}\widehat{\boldsymbol{\beta}} = \mathbf{X}^T\mathbf{y},$$

which is known as the normal equation. Thus (why)

$$\widehat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}$$

$\widehat{\boldsymbol{\beta}}$ is called the least squares estimate of $\boldsymbol{\beta}$. However, finding inverse is computationally not stable!!

Computing with data

Question What is a stable algorithm?

Computing with data

Question What is a stable algorithm?

Floating point number system A floating point number system $F \subset \mathbb{R}$ is a set whose elements have the form

$$y = \pm m \times \beta^{e-t}$$

the *base* β (sometimes called the radix)

Computing with data

Question What is a stable algorithm?

Floating point number system A floating point number system $F \subset \mathbb{R}$ is a set whose elements have the form

$$y = \pm m \times \beta^{e-t}$$

the *base* β (sometimes called the radix)

the *precision* t

Computing with data

Question What is a stable algorithm?

Floating point number system A floating point number system $F \subset \mathbb{R}$ is a set whose elements have the form

$$y = \pm m \times \beta^{e-t}$$

the *base* β (sometimes called the radix)

the *precision* t

the *exponent range* $e_{\min} \leq e \leq e_{\max}$

Computing with data

Question What is a stable algorithm?

Floating point number system A floating point number system $F \subset \mathbb{R}$ is a set whose elements have the form

$$y = \pm m \times \beta^{e-t}$$

the *base* β (sometimes called the radix)

the *precision* t

the *exponent range* $e_{\min} \leq e \leq e_{\max}$

the *significand* (sometimes called mantissa) is an integer satisfying $0 \leq m \leq \beta^t - 1$

Computing with data

Question What is a stable algorithm?

Floating point number system A floating point number system $F \subset \mathbb{R}$ is a set whose elements have the form

$$y = \pm m \times \beta^{e-t}$$

- the *base* β (sometimes called the radix)
- the *precision* t
- the *exponent range* $e_{\min} \leq e \leq e_{\max}$
- the *significand* (sometimes called mantissa) is an integer satisfying $0 \leq m \leq \beta^t - 1$
- to ensure a unique representation for each nonzero $y \in F$, it is assumed that $m \geq \beta^{t-1}$ (normalized) if $y \neq 0$ (0 is a special case)

Computing with data

Question What is a stable algorithm?

Floating point number system A floating point number system $F \subset \mathbb{R}$ is a set whose elements have the form

$$y = \pm m \times \beta^{e-t}$$

- the *base* β (sometimes called the radix)
- the *precision* t
- the *exponent range* $e_{\min} \leq e \leq e_{\max}$
- the *significand* (sometimes called mantissa) is an integer satisfying $0 \leq m \leq \beta^t - 1$
- to ensure a unique representation for each nonzero $y \in F$, it is assumed that $m \geq \beta^{t-1}$ (normalized) if $y \neq 0$ (0 is a special case)

Computing with data

The range of floating point numbers in F is :

$$\beta^{e_{\min}-1} \leq |y| \leq \beta^{e_{\max}}(1-\beta^{-t})$$

Computing with data

The range of floating point numbers in F is :

$$\beta^{e_{\min}-1} \leq |y| \leq \beta^{e_{\max}}(1-\beta^{-t})$$

Note An alternative way of expressing y is

$$y = \pm\beta^e \left(\frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \ldots + \frac{d_t}{\beta^t}\right) = \pm \underbrace{.d_1 d_2 \ldots d_t}_{t\text{-digit fraction}} \times \beta^e$$

where $0 \leq d_i \leq \beta - 1$ and $d_1 \neq 0$ (for normalized numbers)

Computing with data

The range of floating point numbers in F is :

$$\beta^{e_{\min}-1} \leq |y| \leq \beta^{e_{\max}}(1-\beta^{-t})$$

Note An alternative way of expressing y is

$$y = \pm\beta^{e}\left(\frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \ldots + \frac{d_t}{\beta^t}\right) = \pm \underbrace{.d_1 d_2 \ldots d_t}_{t\text{-digit fraction}} \times \beta^{e}$$

where $0 \leq d_i \leq \beta - 1$ and $d_1 \neq 0$ (for normalized numbers) In this representation, d_1 is called the *most significant digit* and d_t the *the least significant digit*

Computing with data

The range of floating point numbers in F is :

$$\beta^{e_{\min}-1} \leq |y| \leq \beta^{e_{\max}}(1-\beta^{-t})$$

Note An alternative way of expressing y is

$$y = \pm\beta^{e}\left(\frac{d_1}{\beta}+\frac{d_2}{\beta^2}+\ldots+\frac{d_t}{\beta^t}\right) = \pm \underbrace{.d_1 d_2 \ldots d_t}_{t\text{-digit fraction}} \times \beta^{e}$$

where $0 \leq d_i \leq \beta - 1$ and $d_1 \neq 0$ (for normalized numbers) In this representation, d_1 is called the *most significant digit* and d_t the *the least significant digit*

Observation Floating points are not equally spaced. Set $\beta = 2$, $t = 3$, $e_{\min} = -1$, $e_{\max} = 3$