Computing: from classical to quantum

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  Message = information (??)

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Main elements of a Turing machine M:

1. (scratch pad) k tapes: each is infinite and divided into cells, each cell holds one letter $a \in \Gamma = \{0, 1, \square, \triangleright\}$, called the alphabet of M. Each tape is quipped with a head that can read or write letters to the tape one cell at a time. The first tape is read-only, the input tape and the $k-1$ tapes are read-write, called the work tapes. The last one is the output tape, on which it writes the final answer

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2. A control unit/register: a finite number of possible states $Q = \{q_s, q_1, \ldots, q_l, q_h\}$, q_s and q_h are the start state and the halting state, respectively. The state determines its action at the next computational step:
   (i) real the letters
   (ii) for the $k-1$ read-write tapes, replace each letter with a new letter
   (iii) change its register to contain another state from Q
   (iv) move each head one cell to left or right or stay at the same place

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Program: a finite set of instructions for each tape

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3. the displacement of the read/write head one cell left or right or stay

Three functions

$$q_j = f_q(q_i, a_k) \tag{1}$$
$$a_l = f_a(q_i, a_k) \tag{2}$$
$$d = f_d(q_i, a_k), \tag{3}$$

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Question Does a TM halt at every input in a finite number of steps?

Turing machine

Thus the working of a Turing machine at each tape is described by

$$(q_i, a_k) \mapsto (q_j, a_l, d)$$

Question Does it have any resemblance in mordern day computers?

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Computing a function and running time[1] Let $f : \{0,1\}^* \to \{0,1\}^*$ and let $T : \mathbb{N} \to \mathbb{N}$ be some function, and let M be a Turing machine. We say that M computes f if for every $x \in \{0,1\}^*$, whenever M is initialized to the start configuration on input x, then it halts with $f(x)$ written on its output tape. We say M computes f in $T(n)$-time if its computation on every input x requires at most $T(|x|)$ steps.

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- The universal Turing machine
- The probabilistic Turing machine
- The halting problem (undecidable!!)

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For instance: a positive integer $N < 2^n$ can be written as

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and hence equivalently

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The binary codes for non-integer numbers:

$$5.5 = 101.1,\ 5.25 = 101.01,\ 5.125 = 101.001$$

Circuit model of computation

- The advantage of binary numbers is that they can be stored in electrical devices with two possible values - such as high and low voltages or switches with only two positions on and off can be used to load one bit of information

Elementary logic gates Logical function with n-bit input and m-bit output:

$$f : \{0,1\}^n \to \{0,1\}^m$$

Universal gates: Any function $f : \{0,1\}^n \to \{0,1\}^m$ can be constructuted from the elementary gates AND, OR, NOT, and COPY. Thus these gates constitute a universal model of computation.

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Computational complexity- find the minimum resources to solve a problem with the best possible algorithm

Notation Given two functions $f(n)$ and $g(n)$, we write $f = O(g)$ if

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An answer In 1971 Schonhage and Strassen discovered an algorithm that requires $O(n \log n \log \log n)$

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Example

1. The best known algorithm for the factorization of an integer N requires $\exp(O(n^{1/3}(\log n)^{2/3}))$ operations, where $n = \log N$.

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2. However, a polynomial algorithm scaling as n^α, $\alpha \gg 1$, like $\alpha = 10^3$ can hardly be regarded be easy

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Observation Shor's quantum algorithm with polynomial resource can solve the factorization problem, however if such a classical algorithm does not exist then only we will be able to say that quantum model of computation is powerful than classical!!

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Question Boolean functions and languages are equivalent!!

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Question Under what condition $\mathbf{P} = \mathbf{NP}$ or $\mathbf{P} \neq \mathbf{NP}$
Note The factorization problem and graph isomorphism problem are not known to be in **P** nor **NPC**

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Reduction, **NP**-hardness and **NP**-completeness A language L is polynomial-time reducible to a language L', denoted as $L \leq_p L'$, if there is a polynomial-time computable function such that for every input x, $x \in L$ if and only if $f(x) \in L'$. Then we say

L' is **NP**-hard if $L \leq_p L'$ for every $L \in$ **NP.**

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Question Can you explain **NP**-hard languages in one line?

Question Why is the notion of **NPC** significant?

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- **BPP** - a decision problem is in in this class if there exists a polynomial-time algorithm (in a probabilistic Turing machine) such that the probability of getting the right answer is larger than $\frac{1}{2} + \delta$ for every possible input and $\delta > 0$
- **BQP** - a decision problem is in this class if there is a polynomial-time quantum algorithm that gives the right answer with probability larger than $\frac{1}{2} + \delta$, $\delta > 0$.
  Example Shor's algorithm belongs to this class with $O(n^2 \log n \log \log n \log(1/\epsilon))$, ϵ is the probability of error.

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Question Can we say that a quantum computer would be better than a classical computer?

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Note The circuits in silicon chips used in modern computers are not acyclic and use cycles to implement memory. However, any computation that runs on a silicon chip with g gates and finishes in time t, can also be performed by a Boolean circuit of size $O(gt)$

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Note The circuits in silicon chips used in modern computers are not acyclic and use cycles to implement memory. However, any computation that runs on a silicon chip with g gates and finishes in time t, can also be performed by a Boolean circuit of size $O(gt)$

Homework Uniform vs non-uniform models

Boolean circuits

A $T(n)$-size circuit family is a sequence $\{C_n\}_{n\in\mathbb{N}}$ of Boolean circuits, where C_n has n inputs and single output, and its size $\leq T(n)$ for every n.

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Homework Depth complexity of a Boolean function

Quantum computation

Question TM and Circuit models are equivalent!!

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Recall The Shor's polynomial-time quantum algorithm for factorizing integers pose a serious challenge to the strong Church-Turing thesis since no polynomial time algorithm is known for deterministic or probabilistic Turing machines. Thus if quantum computers are physically realizable then the strong Church-Turing thesis is wrong.

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Note[2] TM fails to capture all physically realizable computing devices for a fundamental reason: the TM is based on a classical physics model of the universe, whereas current physical theory asserts that the universe is quantum physical.

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Quantum computation models

- Configuration of a TM - complete description of the contents of the tape, the location of the tape head, and the state $q \in Q$ of the control

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- Consequently, if a probability distribution is represented as $|v\rangle$ then the distribution at the next step is $M|v\rangle$
- M is refereed to as 1time evolution operator'

Quantum Turing machine[3]

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Quantum Turing machine (QTM)

Let $\widetilde{\mathbb{C}}$ denote the set of complex numbers α such that there is a deterministic algorithm that computes the real and imaginary parts of α to within 2^{-n} in time polynomial in n.

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Then a QTM[4] (single tape) is a triplet (Γ, Q, d) with the quantum transition function

$$\delta : Q \times \Gamma \to \widetilde{\mathbb{C}}^{\Gamma \times Q \times \{L,R\}}$$

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Let S be the inner product space of finite linear combinations of configurations with the Euclidean norm. Then QTM M defines a linear operator $U_M : \ S \to S$: if M starts in configuration c with current state p and scanned symbol σ, then after one step M will be in superposition of configurations $\psi = \sum_i \alpha_i \, c_i$, where α_i corresponds to the transition $\delta(p, ...)$, and c_i is the new configuration that results from applying this transition c. Extending this map to the entire space S through linearity gives the liner time evolution operator U_M

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- A quantum circuit on n qubits implements a unitary transformation on the Hilbert space $(\mathbb{C}^2)^{\otimes n}$
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$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, R(\delta) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{bmatrix}$$

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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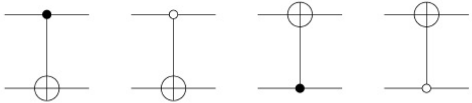
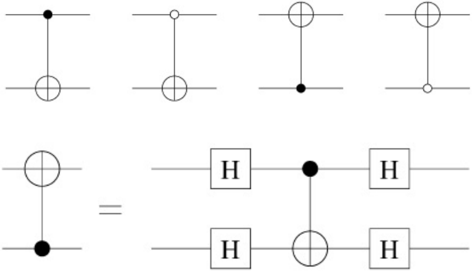
Note $CNOT(\alpha|0\rangle + \beta|1\rangle)|0\rangle = \alpha|00\rangle + \beta|11\rangle$, which is not separable when $\alpha, \beta \neq 0$.

Quantum gates



Quantum gates



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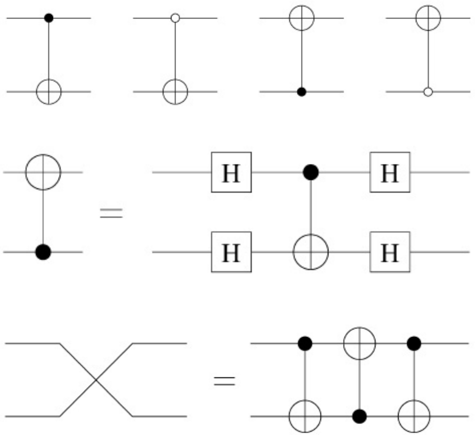
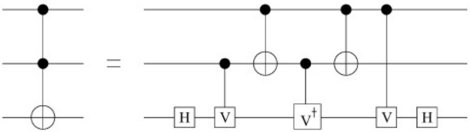


Quantum gates

Toffolli gate: $V = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$



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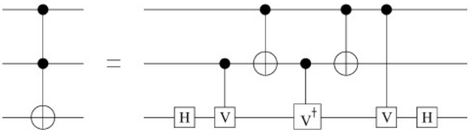
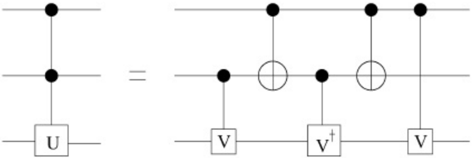


C^2-U gate: $V^2 = U$



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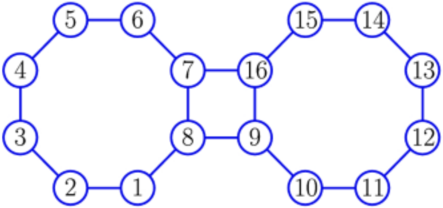
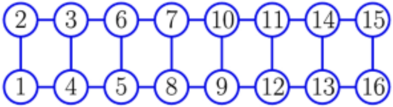
Homework Circuit complexity, Query complexity

Challenge for NISQ computers?

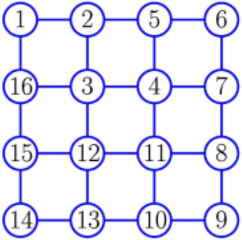
- ▷ limited connectivity between qubits: the coupling constraints



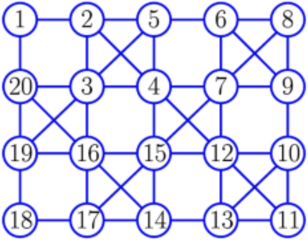
(a) Rigetti 16Q-Aspen



(b) IBM QX5



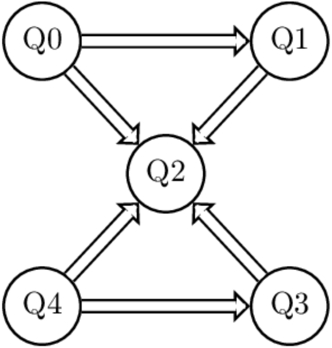
(c) 16q-Square



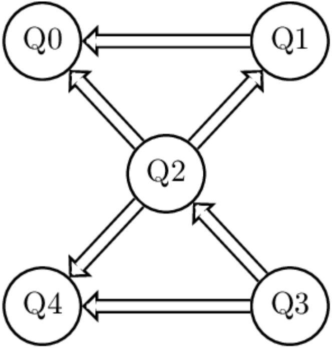
(d) IBM QX20 Tokyo

Challenge for NISQ computers?

- ▷ timespace coordinates



QX2



QX4

Information

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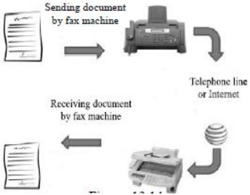
- ▷ Think of a fax machine - the font size of the words



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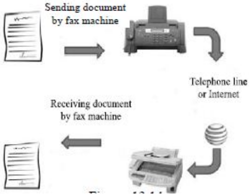

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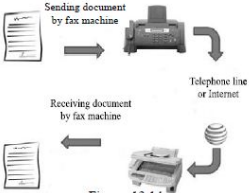

- ▷ Physics of information - how to store and process?
  - ▽ Claude E Shannon (1916 - 2001) - father of information theory
  - ▽ A Mathematical Theory of Communication

Content of the course

Shannon's theory/model

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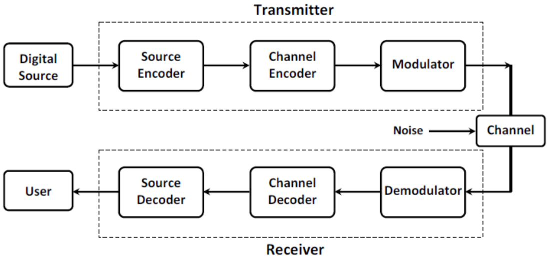
Quantum entropy existed before classical entropy!!

Model of a digital communication system



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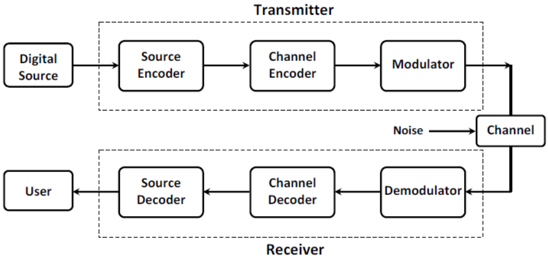

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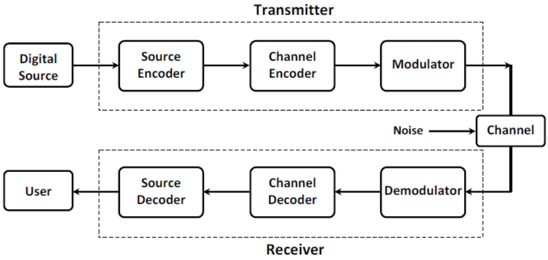

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The encoding paradigm: Here

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Question How much surprised you are if India wins in a football match against Argentina?

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Proof Let Y be a rv which takes the value $1/p(x)$ with probability $p(x)$. Then

$$\sum_{x\in\mathcal{X}} p(x) \cdot \log\left(\frac{1}{p(x)}\right) = \mathbb{E}[\log(Y)] \leq \log(\mathbb{E}[Y])$$

Entropy

Entropy Suppose X is a rv distributed over $\mathcal{X} = \{a_1, \ldots, a_n\}$ such that each value $x \in \mathcal{X}$ occurs with probability $p(x)$. Then the entropy of X is

$$H(X) = \sum_{x\in\mathcal{X}} p(x) \cdot \underbrace{\log_2\left(\frac{1}{p(x)}\right)}_{\text{surprise}} = -\sum_{x\in\mathcal{X}} p(x) \cdot \log_2(p(x)) \text{ bits}$$

Example Consider the entropy of coin toss with p as probability of head. What happens if $p_i = 1/n$?

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Example $\Sigma = \{0, 1\}$. Let $\mathcal{X} = \{a, b, c, d\}$ with $p(a) = 1/2$, $p(b) = 1/4$, $p(c) = 1/8$ and $p(d) = 1/8$. How do we design a code for $\mathcal{X}$ such that expected length of the code is minimized?

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For any alphabet Σ, replace 2^{l_i} by $|\Sigma|^{l_i}$.

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Answer $a = 0$, $b = 10$, $c = 110$, $d = 111$

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the expected number of bits used is

$$\sum_{x \in \mathcal{X}} p(x) \cdot \lceil \log(1/p(x)) \rceil \leq \sum_{x \in \mathcal{X}} p(x) \cdot (\log(1/p(x)) + 1) = H(X) + 1.$$

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Chain rule of entropy

Set $H(Y|X) = \mathbb{E}_x[H(Y|X = x)]$. Then we have

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Homework Let (X, Y) be a joint random variable with $X \vee Y = 1$, $X \in \{0, 1\}$ and $Y \in \{0, 1\}$ such that $p(0, 1) = p(1, 0) = p(1, 1) = 1/3$. Then calculate $H(X)$, $H(Y)$, $H(Y|X = 0)$, $H(Y|X = 1)$, $H(Y|X)$, $H(X, Y)$

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Proposition $H(Y) \geq H(Y|X)$

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Proof

$$H(Y|X) - H(Y) = \sum_x p(x) \sum_y p(y|x) \log \frac{1}{p(y|x)} - \sum_y p(y) \log \frac{1}{p(y)}$$

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&= \sum_{x,y} p(x,y) \left(\log \frac{p(x)p(y)}{p(x,y)} \right)
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Entropy

Now let W be a rv that takes the value $\frac{p(x)p(y)}{p(x,y)}$ with probability $p(x, y)$. Then using jensen's inequality

$$\sum_{x,y} p(x, y) \left(\log \frac{p(x)p(y)}{p(x, y)} \right) \leq \log \left(\sum_{x,y} \frac{p(x)p(y)}{p(x, y)} p(x, y) \right) = \log(1) = 0$$

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Homework $H(Y|X, Z) \leq H(Y|Z)$

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General case Suppose $\overline{X} = (X_1, X_2, \ldots, X_m)$.

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Homework Show (by induction) that

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Question Can the upper bound for expected code length of $H(X) + 1$ be improved?

Entropy

Recall

- ▷ Let X be a rv with range set $\{a_1, \ldots, a_n\}$ and $p(a_i) = p_i$

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- ▷ If $l_1, l_2, \ldots, l_n$ are the codeword lengths for $a_1, \ldots, a_n$ respectively then

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- We proved that the expected length is bounded below by $H(X)$ and bounded above by $H(X) + 1$ (Shannon code)

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Recall

- Let X be a rv with range set $\{a_1, \dots, a_n\}$ and $p(a_i) = p_i$
- We want to encode a_is with expected code length small i.e. expected number of bits needed is small
- If $l_1, l_2, \dots, l_n$ are the codeword lengths for $a_1, \dots, a_n$ respectively then

$$\sum_{i=1}^{n} 2^{l_i} \leq 1$$

- We proved that the expected length is bounded below by $H(X)$ and bounded above by $H(X) + 1$ (Shannon code)

Question Can we improve the upper bound?

Entropy

The idea - Source Coding Theorem

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- Then

$$\begin{aligned} H(X_1, \ldots, X_m) &= H(X_1) + H(X_2|X_1) + \ldots + H(X_m|X_1, \ldots, X_{m-1}) \\ &= H(X_1) + H(X_2) + \ldots + H(X_m) \\ &= m \cdot H(X) \end{aligned}$$

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Theorem (Fundamental Source Coding Theorem (Shannon)). For any $\epsilon > 0$ there exists a n_0 such that for all $n \geq n_0$ and given n copies of X, $X_1, \ldots, X_n$ sampled i.i.d., it is possible to communicate $(X_1, \ldots, X_n)$ using at most $H(X) + \epsilon$ bits per copy on average.

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Homework Let X, Y be two variables with $X \vee Y = 1$, $X \in \{0,1\}$, $Y \in \{0,1\}$ such that $(X,Y) = (1,0)$, $(X,Y) = (0,1)$ and $(X,Y) = (1,1)$ with probabilities $1/3$. Then calculate $I(X;Y)$

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Conditional mutual information

$$I(X;Y|Z) \quad = \quad \mathbb{E}_Z[I(X|Z=z;Y|Z=z)]$$

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I(X;Y|Z) &= \mathbb{E}_Z[I(X|Z=z;Y|Z=z)] \\
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Example Let (X, Y, Z) be a random variable with $Z = X \oplus Y$, $X \in \{0,1\}$, $Y \in \{0,1\}$ such that $(X, Y, Z) = (x, y, z)$ are equally likely. Then check that $I(X;Y) = 0$ and

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$$\begin{aligned} D(P\|Q) &= \frac{2}{3}\log\frac{2}{3} + \infty = \infty \\ D(Q\|P) &= \log\frac{3}{2} + 0 = \log\frac{3}{2} \end{aligned}$$

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$$\begin{aligned} D(P\|Q) = \sum_x p(x) \log \frac{p(x)}{q(x)} &= \sum_{x \in \mathfrak{Supp}(P)} p(x) \log \frac{p(x)}{q(x)} \\ &\geq -\log \left(\sum_{x \in \mathfrak{Supp}(P)} p(x) \cdot \frac{q(x)}{p(x)} \right) \\ &= -\log \left(\sum_{x \in \mathfrak{Supp}(P)} q(x) \right) \geq -\log 1 = 0 \end{aligned}$$

Kl divergence

Interpretation of KL divergence in terms of source coding

$$D(P\|Q) = \sum_x p(x)\log\frac{p(x)}{q(x)} = \sum_x p(x)\log\frac{1}{q(x)} - \sum_x p(x)\log\frac{1}{p(x)}$$

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- $\rightarrow$ This can be interpreted as the number of extra bits we use (on average) if we designed a code according to the distribution P, but used it to communicate outcomes of a random variable X distributed according to Q
- $\rightarrow$ The first term in the RHS, which corresponds to the average number of bits used by the "wrong" encoding, is also referred to as *cross entropy*

Code

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Extension of a code The extension C^* of a code C is the mapping from the finite strings of $\mathcal{X}$ to finite strings of the alphabet set i.e.

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Uniquely decodable code A code is uniquely decodable if its extension is nonsingular i.e. any encoded string is a uniquely decodable code has only one possible source string

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Question Can we construct a uniquely decobale code with expected codeword length $H(X)$? - optimal codeword length (Huffman code)

Channel capacity

$$\underbrace{(x_1, x_2, \ldots, x_n)}_{\text{input}} \rightarrow \underbrace{(y_1, y_2, \ldots, y_n)}_{\text{output}}$$

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$p(y_j|x_k)$ are called transition probabilities

Channel capacity

Memoryless channel if each output letter in the output sequence depends only on the corresponding in put i.e.

$$p_N(\overline{y}|\overline{x}) = p_N((y_1 \dots y_N)|(x_1 \dots x_N)) = \prod_{n=1}^{N} p(y_n|x_n)$$

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Example Binary symmetric channel

Channel capacity

Alternative interpretation of mutual information Suppose

$$I(x; y) = \log \frac{p(x|y)}{p(x)} = \log \frac{p(y|x)}{p(y)} = \log \frac{p(x, y)}{p(x)p(y)} = I(y; x)$$

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Homework $I(X;Y) = H(X) - H(X|Y)$!!
Question Does it have any connection with the KL-divergence?

Channel capacity

The largest 'average' mutual information that can be obtained over the channel

$$C = \max_{p(X)} I(X;Y)$$

i.e.

$$\max I(X;Y) \text{ wrt } \sum_{k=1}^{K} p_k = 1, p_k \geq 0$$

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Question Does it exist?

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Theorem (DMC) Let $\overline{X}^N, \overline{Y}^N$ denote the random variables corresponding to the sequences of N-length input and output sequences respectively:

$$\overline{X}^N = (X_1, \ldots, X_N),\ \overline{Y}^N = (Y_1, \ldots, Y_N),$$

where X_i, Y_i are iid. Then

$$I(\overline{X}^N; \overline{Y}^N) \leq \sum_{n=1}^{N} I(X_n; Y_n)$$

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Question What is the conclusion of this theorem?

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→ The source encoder transform the source output into a string of bits, called the information sequence

- △ The number of bits per unit time required to represent the source output is minimized
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The big picture

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Problem Design and implementation of encoder/decoder pair such that - information can be transmitted in noisy environment, and the information can be reliably reproduced at the output of the channel decoder

Codes

Observation

→ The k-tuple $\mathbf{u} = (u_0, u_1, \ldots, u_{k-1})$, called a message (sometimes $\mathbf{u}$ is used to denote a k-bit message rather than the entire information sequence)

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- → The ratio $R = k/n$ is called the *code rate*, and it can be interpreted as the number of information bits entering the encoder per transmitted symbol
- → Each message is encoded independently, so the encoder is memoryless and can be implemented with a *combinatorial logic circuit*

Linear block codes

Definition A block code of length n and 2^k codewords is called a linear (n, k)-code if and only if its 2^k codewords form a k-dimensional subspace of the vector space of all n-tuples over the field $GF(2)$, the Galois Field of order 2

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Conclusion

- △ A binary block code is linear if and only if the modulo-2 sum of two codewords is also a codeword
- △ Since (n, k) linear block code C is a k-dimension subspace of V_n, the vector space of all binary n-tuples, it is possible to find k linearly independent codewords $\mathbf{g}_0, \mathbf{g}_1, \ldots, \mathbf{g}_{k-1}$ in C such that any codeword $\mathbf{v}$ in C can be written as

$$v = u_0\mathbf{g}_0 + u_1\mathbf{g}_1 + \ldots u_{k-1}\mathbf{g}_{k-1}$$

where $u_i \in \{0, 1\}, 0 \leq i \leq k-1$

Linear block codes

Write

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}_0 \\ \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_{k-1} \end{bmatrix} = \begin{bmatrix} g_{00} & g_{01} & \cdots & g_{0,n-1} \\ g_{10} & g_{11} & \cdots & g_{1,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ g_{k-1,0} & g_{k-1,1} & \cdots & g_{k-1,n-1} \end{bmatrix}_{k \times n}$$

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Then

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Example

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}_0 \\ \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

generates a $(7, 4)$ linear code

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Question Verify that $\mathbf{v} = (0\,0\,0\,1\,1\,0\,1)$ is a codeword for the above generator matrix

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Systematic format of a codeword A codeword is divided into two parts - the message part and the redundant checking part

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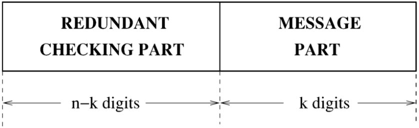
The message part consists of k unaltered information digits, and the redundant checking part consists of $n - k$ *parity-check* digits



A linear block with this structure is referred to as *linear systematic block code*

Linear block code

Thus a linear systematic (n, k) code is completely described by a $k \times n$ matrix $\mathbf{G}$ of the following form

$$\mathbf{G} = \begin{bmatrix} \mathbf{P} & I_k \end{bmatrix}, \ \mathbf{P} = [p_{ij}] \in \{0, 1\}^{k \times (n-k)}$$

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Let $\mathbf{u} = (u_0, u_1, \ldots, u_{k-1})$ be the message to be encoded. Then the corresponding codeword is

$$\mathbf{v} = \mathbf{u} \cdot \mathbf{G}$$

which gives two equations

$$\begin{aligned} v_{n-k+i} &= u_i, \ 0 \leq i \leq k-1 && (5) \\ v_j &= u_0 p_{0j} + u_1 p_{1j} + \ldots + u_{k-1} p_{k-1,j}, \ 0 \leq j \leq n-k-1. && (6) \end{aligned}$$

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The $(n-k)$ equations given by equation (6) are called parity-check equations.

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Then an n-tuple $\mathbf{v}$ is a codeword in the code C generated by $\mathbf{G}$ if and only if $\mathbf{v} \cdot \mathbf{H}^T = \mathbf{0}$

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Note The rows of $\mathbf{H}$ also generate a $(n, n-k)$ linear code C_d, which is called the dual code of C.

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Then the code C is just the null-space of $\mathbf{H}$, which is called a parity-check matrix of the code.

Note The rows of $\mathbf{H}$ also generate a $(n, n-k)$ linear code C_d, which is called the dual code of C.

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If the generator matrix of an (n, k) linear code is in the systematic form then the parity-check matrix can be in the following form:

$$\mathbf{H} = \begin{bmatrix} I_{n-k} & \mathbf{P}^T \end{bmatrix}.$$

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Then see that

$$\mathbf{G} \cdot \mathbf{H}^T = \mathbf{0}.$$

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Syndrome decoding Consider an (n, k) linear code corresponding to generator matrix $\mathbf{G}$ and parity-check matrix $\mathbf{H}$. Let $\mathbf{r} = (r_0, r_1, \ldots, r_{n-1})$ be the received vector at the output of a noisy channel corresponding to a codeword $\mathbf{v} = (v_0, v_1, \ldots, v_{n-1})$.

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Then

$$\mathbf{r} = \mathbf{v} + \mathbf{e} \Rightarrow \mathbf{e} = \mathbf{r} + \mathbf{v} = (e_0, e_1, \ldots, e_{n-1})$$

is the *error vector*, where $e_i = 1$ for $r_i \neq v_i$, and $e_i = 0$ for $r_i = v_i$.

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Note The receiver does not know both $\mathbf{v}$ and $\mathbf{e}$

Question How does the receiver detect, locate and correct the error?

Linear block code

On receiving $\mathbf{r}$, the decoder must first determine whether $\mathbf{r}$ contains transmission errors. Thus the decoder computes

$$\mathbf{s} = \mathbf{r} \cdot \mathbf{H}^T = (s_0, s_1, \ldots, s_{n-k-1})$$

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Then $\mathbf{s} = \mathbf{0}$ if and only if $\mathbf{r}$ is a codeword, and $\mathbf{s} \neq \mathbf{0}$ if and only if $\mathbf{r}$ is not a codeword. Thus when $\mathbf{s} = 0$, $\mathbf{r}$ is a codeword, and the receiver accepts $\mathbf{r}$ as the transmitted codeword.

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Caution It is possible that the errors in certain error vectors are not detectable. For instance, if $\mathbf{e}$ is identical to a nonzero codeword. This kind of error patterns are called *undetectable* error patterns. There are $2^k - 1$ undetectable errors (Homework)

Linear block code

However, note that

$$\mathbf{s} = \mathbf{r} \cdot \mathbf{H}^T = (\mathbf{v} + \mathbf{e}) \cdot \mathbf{H}^T = \mathbf{v} \cdot \mathbf{H}^T + \mathbf{e} \cdot \mathbf{H}^T = \mathbf{e} \cdot \mathbf{H}^T$$

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Note that there are $n - k$ linear equations and the system does not have a unique solution but can have 2^k solutions!!

Linear block code

Minimum distance of a block code Let $\mathbf{v} = (v_0, v_1, \ldots, v_{n-1})$ be an n-tuple. Then the *Hamming weight* of $\mathbf{v}$, denotes as $w(\mathbf{v})$ is the number of nonzero entries of $\mathbf{v}$.

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The minimum distance of a code C is defined by

$$d_{\min} = \min\{d_h(\mathbf{v}, \mathbf{w}) : \mathbf{v}, \mathbf{w} \in C, \mathbf{v} \neq \mathbf{w}\}$$

Linear block code

Note that

$$\begin{aligned} d_{\min} &= \min\{w(\mathbf{v}+\mathbf{w}) : \mathbf{v}, \mathbf{w} \in C, \mathbf{v} \neq \mathbf{w}\} \\ &= \min\{w(\mathbf{x}) : \mathbf{x} \in C, \mathbf{x} \neq 0\} \end{aligned}$$

Thus minimum distance of a linear code is the minimum weight of the code.

Linear block code

Theorem Let C be an (n, k) linear code with parity-check matrix $\mathbf{H}$. Then for each codeword of Hamming weight l, there exists l columns of $\mathbf{H}$ such that the sum of these l columns is equal to the zero vector. Conversely, if there exist l columns of $\mathbf{H}$ whose sum is the zero vector then there exists a codeword of Hamming weight l in C.

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(a) If no $d - 1$ or fewer columns of $\mathbf{H}$ add to $\mathbf{0}$, the code has minimum weight at least d

(b) The minimum distance of C is equal to the smallest number of columns of $\mathbf{H}$ that sum to $\mathbf{0}$.

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Error detection and error correction Suppose a codeword $\mathbf{v}$ is transmitted over a noisy channel. Then a block code with minimum distance $d_{\min}$ is capable of detecting all the error patterns of $d_{\min} - 1$ or fewer errors:

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- $\rightarrow$ Then for this code, no error pattern of $d_{\min} - 1$ or fewer errors can change one codeword into another, hence any error pattern of $d_{\min} - 1$ or few errors will result in a received vector $\mathbf{r}$ that is not a codeword in C

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Question Can it detect all the error patterns of $d_{\min}$ errors?

Linear block code

Observation (n, k) linear block code can detect $2^n - 2^k$ error patterns of length n

$\rightarrow$ The number of nonzero error patterns is equal to $2^n - 1$, among which $2^k - 1$ error patterns are the $2^k - 1$ nonzero codewords.

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- $\rightarrow$ Note that there are exactly $2^n - 2^k$ error patterns that are not identical to the codewords of the (n, k) block code, which are detectable
- $\rightarrow$ For large n, $2^k - 1 \ll 2^n$ in general, hence only a small fraction of error patterns pass through the decoder without being detected

Linear block code

Maximum-Likelihood (ML) decoding

→ A decoder must determine **w** to minimize

$$P(E|\,\mathbf{r}) = P(\mathbf{w} \neq \mathbf{v}|\,\mathbf{r})$$

→ The probability of error is

$$P(E) = \sum_{\mathbf{r}} P(E|\,\mathbf{r})P(\mathbf{r})$$

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△ Maximize $P(\mathbf{r}|\,\mathbf{v}) = \prod_j P(r_j|\,v_j)$

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- △ Alternatively, choose $\mathbf{v}$ to maximize $\log P(\mathbf{r}|\,\mathbf{v}) = \sum_j \log P(r_j|\,v_j)$
- △ The ML decoder is optimal if and only if all $\mathbf{v}$ are equally likely as input vectors, otherwise $P(\mathbf{r}|\,\mathbf{v})$ must be weighted by the codeword probabilities $P(\mathbf{v})$

Linear block code

ML decoding on the BSC Suppose the noisy channel is BSC with bit-flip probability ϵ. Then

$\rightarrow$ $P(r_j | v_j) = 1 - \epsilon$ if $r_j = v_j$ and ϵ otherwise

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\begin{aligned}
\log P(\mathbf{r}|\mathbf{v}) &= \sum_j \log P(r_j | v_j) \\
&= d(\mathbf{r},\mathbf{v}) \log \epsilon + (n - d(\mathbf{r},\mathbf{v})) \log(1-\epsilon) \\
&= d(\mathbf{r},\mathbf{v}) \log \frac{\epsilon}{1-\epsilon} + n \log(1-\epsilon)
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→ $\log \frac{\epsilon}{1-\epsilon} < 0$ for $\epsilon < 0.5$, so an ML decoder for a BSC must choose $\mathbf{v}$ to minimize $d(\mathbf{r},\mathbf{v})$

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Then for a linear block code, an ML decoder takes n received bits as input and returns the most likely k-bit message among the 2^k possible messages.

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→ Compare the received word $\mathbf{r}$ to each of these valid codewords and find the one with smallest Hamming distance to $\mathbf{r}$

→ However, it has exponential time complexity. What we would like is something a lot faster. Note that this comparing to all valid codewords method does not take advantage of the linearity of the code.

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for some positive integer t.

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→ Let $\mathbf{w}$ be any other codeword of C. Then

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→ Obviously, $d(\mathbf{v}, \mathbf{w}) \geq d_{\min} \geq 2t + 1$, and hence $d(\mathbf{w}, \mathbf{r}) \geq 2t + 1 - t'$

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$\rightarrow$ If $t' < t$ then $d(\mathbf{w}, \mathbf{r}) > t$

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$\rightarrow$ According to ML decoding scheme, it is a correct transmitted codeword, thus the errors are corrected.

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- Classical information is carried by systems with a definite state, and it can be replicated and measured without being altered

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    effect of quantum entanglement on information transmission
    informational aspect of the quantum measurement process, the trade offs between the disturbance of the quantum state and the accuracy of the measurement

von Neumann entropy

For a density matrix ρ, of an n-qubit system

$$S(\rho) = -tr[\rho \log \rho]$$

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Setting, $\rho = \sum_{j=1}^{2^n} p_i \left|e_j\right\rangle \left\langle e_j\right|$, (spectral decomposition)

$$\log \rho = \sum_{j=1}^{2^n} (\log p_j) \left|e_j\right\rangle \left\langle e_j\right|$$

and hence

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von Neumann entropy

For a density matrix ρ, of an n-qubit system

$$S(\rho) = -tr[\rho \log \rho]$$

Setting, $\rho = \sum_{j=1}^{2^n} p_i \left|e_j\right\rangle \left\langle e_j\right|$, (spectral decomposition)

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Question What does this mean?

von Neumann entropy

Observations

- If $\rho = |\psi\rangle \langle\psi|$, is a pure state then it has only one eigenvalue 1, hence trace is 1 and $S(\rho) = 0$

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- We can safely say that von Neumann entropy is the least amount of information to be used to create ρ, and equivalently we can say that it is the minimum amount of classical information that we can access from ρ
- Consider evolution of a system described by ρ : $\rho(t) = e^{-iHt}\rho e^{iH(t)}$, then $S(\rho(t)) = S(\rho)$ - second law of thermodynamics, the entropy of a closed system never decreases

von Neumann entropy

Let ρ_{AB} denote a 'joint' density matrix corresponding to a bipartite/composite system. Then

$$\rho_A = tr_B(\rho_{AB}),\ \rho_B = tr_A(\rho_{AB})$$

are partial traces of ρ_{AB}

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Note Conditioning cannot increase entropy

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mutual information: for a pair of systems A, B

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Question Are these generalizations of classical relative entropy? Which one to choose?

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Justification:

$$S(\rho_1 \| \rho_2) = \lim_{\epsilon \to 0} S(\rho_1 \| \rho_2 + \epsilon I)$$

Quantum information processing

von Neumann entropy - is it related to the fundamental limit of compression?

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Quantum information processing

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- A simple model of quantum information source[5] is an ensemble of quantum states $\{p_X(x), |\psi_x\rangle\}$ - the source outputs the state $|\psi_x\rangle$ with probability $p_X(x)$

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Question How can we use a quantum channel? What is a noiseless quantum channel?

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Quantum information processing

Alice's State preparation the information source outputs a sequence $|\psi_{x^n}\rangle_{A^n}$ of quantum states according to the ensemble $\{p_X(x), |\psi_x\rangle\}$, where

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Alice can think about purification of the density operator as

$$|\phi_\rho\rangle_{RA} = \sum_x \sqrt{p_X(x)}\, |x\rangle_R |\psi_x\rangle_A ,$$

where R is the lebel for the inaccessible reference system, hence the resulting iid state is $|\psi_\rho\rangle_{RA}^{\otimes n}$

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Encoding Alice encodes the systems A^n according to a compression channel $\mathcal{E}_{A^n \to W}$, where W is a quantum system of dimension 2^{nR}, where R is the rate of the compression

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Decoding Bob sends the system W through a decompression channel $\mathcal{D}_{W \to \widehat{A^n}}$

Quantum information processing

The protocol has ϵ-error if

$$\frac{1}{2}\left\|(|\phi_\rho\rangle_{RA})^{\otimes n} - (\mathcal{D}_{W\to\widehat{A^n}} \circ \mathcal{E}_{A^n\to W})(|\phi_\rho\rangle)_{RA}^{\otimes n}\right\|_1 \leq \epsilon$$

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- a quantum compression rate is achivable is there exists an $(n, R+\delta, \epsilon)$ quantum compression code for all $\delta > 0$, $\epsilon \in (0,1)$, for sufficiently large n
- The quantum data compression limit of ρ is equal to the infimum of all achievable quantum compression rates

Schumacher compression

Data compression theorem Suppose ρ is the density matrix corresponding to a quantum information source. then the von Neumann entropy is equal to the quantum data compression limit of ρ

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Positive map A linear map $\mathcal{M} : \mathcal{L}(H_A) \to \mathcal{L}(H_B)$ is positive if $\mathcal{M}(X_A)$ is positive semi-definite for all positive semi-definite $X_A \in \mathcal{L}(H_A)$

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Complete positivity A linear map $\mathcal{M} : \mathcal{L}(H_A) \to \mathcal{L}(H_B)$ is completely positive if $ID_m \otimes \mathcal{M}$ is a positive map

Quantum channel

Example Unitary evolution is a special kind of quantum channel. Under the action of a unitary channel $\mathcal{U}$, the state evolves as

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The Holevo bound an upper bound of the accessible information in a quantum measurement

Thanks for your attention!!

For questions or comments: bibhas.adhikari AT gmail DOT com