IIT Kharagpur
TS70006: Quantum Mechanics and Quantum Computing
Quantum Computing Assignment - 4

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1. The NOT operator takes $|0\rangle \to |1\rangle$ and $|1\rangle \to |0\rangle$. Find its matrix representation with respect to the basis $\{|+\rangle, |-\rangle\}$, where $|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ and $|-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$.

2. Let $R(\alpha,\beta,\gamma;\theta) = \cos\frac{\theta}{2}I + i\,\sin\frac{\theta}{2}(\alpha\sigma_x + \beta\sigma_y + \gamma\sigma_z)$ where $\alpha,\beta,\gamma,\theta$ are real numbers with $\alpha^2+\beta^2+\gamma^2 = 1$.
   (a) Let $x = (x_1,\, x_2\, x_3)$ be a point on the unit sphere. Explain where does $R(\alpha,\beta,\gamma;\theta)x$ lie on the unit sphere.
   (b) Prove that any unitary matrix U of order 2×2 can be written as $U = \exp(i\eta)R(\alpha,\beta,\gamma;\theta)$ for some real numbers $\alpha,\beta,\gamma,\theta,\eta$.
   (c) Determine the values of $\alpha,\beta,\gamma,\theta,\eta$ when $U = H$, the Hadamard gate.

3. A rotation matrix by an angle γ is given by $R(\gamma) = \begin{bmatrix}\cos(\gamma) & \sin(\gamma)\\ -\sin(\gamma) & \cos(\gamma)\end{bmatrix}$. Describe the action of this operator on a qubit $|\psi\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$.

4. Describe the action of the phase shift gate when considering the Bloch sphere representation of a qubit.

5. A Hadamard gate is applied to the qubit $|\psi\rangle = \cos\theta\,|0\rangle + e^{i\phi}\sin\theta\,|1\rangle$, and subsequently a measurement is made. What is the probability that this measurement finds the system in the state $|1\rangle$?

6. Prove that if an operator U is unitary and Hermitian, then $\exp(i\theta U) = \cos\theta I - i\sin\theta U$.

7. Using Dirac notation write down the outer product representation of the following operators

$$CNOT = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}, CH = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\0&0&\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\end{bmatrix}, CZ = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}.$$

8. Using Dirac notation, find the action of the controlled NOT gate when the control bit is $|1\rangle$ and the target qubit is given by $|0\rangle, |1\rangle$ and $\alpha\,|0\rangle + \beta\,|1\rangle$.

9. Construct a quantum circuit to add two-bit numbers x and y modulo 4. Thus the circuit performs $|x,y\rangle \mapsto |x, x+y \bmod 4\rangle$.

10. Describe a quantum circuit generating the two qubit cluster state $|00\rangle + |01\rangle + |10\rangle - |11\rangle$. Also, prove that the cluster state is entangled.

11. Construct a quantum circuit using CNOT and signle qubit gates that implements the following unitary matrix.

$$\frac{1}{2}\begin{bmatrix}1&1&1&1\\1&i&-1&-i\\1&-1&1&-1\\1&-i&-1&i\end{bmatrix}$$

12. (Award question) Suppose U is a single qubit unitary operation. Find a circuit containing $O(n^2)$ Toffoli, CNOT and single qubit gates which implements $C^n(U)$ gate, $n > 3$, using no work qubits.