IIT Kharagpur
TS70006: Quantum Mechanics and Quantum Computing
Quantum Computing Assignment - 3

Instructor : Dr. Bibhas Adhikari

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1. Let $|\psi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$. Explicitly calculate $|\psi\rangle^{\otimes 2}$ and $|\psi\rangle^{\otimes 3}$.
2. Compute the tensor product $\sigma_x \otimes \sigma_x$ and $\sigma_x \otimes I$.
3. A qubit is in the state $|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle$. A measurement with respect to σ_y is made. Given that the eigenvalues of the σ_y matrix are ± 1, determine the probability that the measurement result is $+1$ and the probability that the measurement result is -1.
4. A system is in the state $|\psi\rangle = \frac{1}{\sqrt{6}}|0\rangle + \frac{\sqrt{5}}{\sqrt{6}}|1\rangle$. A measurement is made with respect to the observable σ_x. What is the expectation or average value?
5. Describe the action of the operators $P_0 \otimes I$ and $I \otimes P_1$ on the state $|\psi\rangle = \frac{|01\rangle-|10\rangle}{\sqrt{2}}$, where $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$, respectively.
6. A system is in the state $|\psi\rangle = \frac{1}{\sqrt{8}}|00\rangle + \frac{\sqrt{3}}{\sqrt{8}}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$.
   (a) What is the probability that measurement finds the system in the state $|\phi\rangle = |01\rangle$?
   (b) What is the probability that measurement finds the first qubit in the state $|0\rangle$? What is the state of the system after measurement?
7. A three-qubit system is in the state $|\psi\rangle = \frac{\sqrt{2}+i}{\sqrt{20}}|000\rangle + \frac{1}{\sqrt{2}}|001\rangle + \frac{1}{\sqrt{10}}|011\rangle + \frac{i}{2}|111\rangle$.
   (a) What is the probability that the system is found in the state $|000\rangle$ if all 3 qubits are measured?
   (b) What is the probability that a measurement on the first qubit only gives $|0\rangle$? What is the post-measurement state of the system?
8. A qubit is in the state $|\psi\rangle = |1\rangle$. A measurement of σ_x is made. What are the matrix representations of the projection operators corresponding to measurement results ± 1? What are the probability of finding measurement results ± 1?
9. A system is in the state $|\psi\rangle = \frac{1}{\sqrt{100}}|00\rangle + \frac{\sqrt{99}}{\sqrt{100}}|11\rangle$. What is the probability that measurement finds the system in the state $|\phi\rangle = |01\rangle$?
10. Prove the following.
    (a) $|\psi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}} \neq |\psi_1\rangle \otimes |\psi_2\rangle$ for some quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$.
    (b) tensor product of two unitary matrices is unitary.
    (c) tensor product of two Hermitian matrices is Hermitian.
    (d) tensor product of two positive semi-definite matrices is positive semi-definite.
    (e) tensor product of two projectors is a projector.
11. Verify the commutator relations $[\sigma_x, \sigma_y] = 2i\sigma_z$, $[\sigma_y, \sigma_z] = 2i\sigma_x$, $[\sigma_z, \sigma_x] = 2i\sigma_y$.
12. Let P_1 and P_2 be two projection operators. Show that, if their commutator $[P_1, P_2] = 0$, then their product $P_1 P_2$ is also a projection operator.

13. Assume that the observable σ_x and σ_y are measured within a system in the state $|0\rangle$, which is the eigenstate of σ_z with respect to the eigenvalue 1. Prove that the uncertainty principle indicates $\Delta\sigma_x\Delta\sigma_y \geq 1$.

14. (Award question) Write the computer programs for the following in your favorite programming language (use of any library made for quantum computing is strictly restricted)

    (a) Given an orthonormal basis, say $\{|0\rangle, |1\rangle\}$ of $\mathbb{C}^2$, the program generates the (prduct) basis vectors of $(\mathbb{C}^2)^{\otimes n}$, call it $|u_1\rangle, \ldots, |u_{2^n}\rangle$ for a given n. Then the program generates a random unit vector $c = [c_1, c_2, \ldots, c_{2^n}] \in (\mathbb{C}^2)^{\otimes n}$, and finally the program produces the state vector $\sum c_i |u_i\rangle$.

    (b) Write a computer program which accepts a quantum state and a measurement operator as an input as well as returns the measurement value and post measurement state as an output.