Indian Institute of Technology Kharagpur
Information and Coding Theory (MA41024/MA60020)
Assignment - 2, Spring Semester 2019-20

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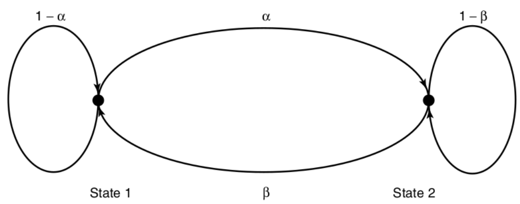
1. Consider a two-state Markov chain with a probability transition matrix $\begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$ as shown in the following figure.



   (a) Prove that the entropy of state X_n at time n is $H(X_n) = H\left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$.

   (b) The entropy rate of the two-state Markov chain in the above figure is given by
$$H(\mathcal{X}) = H(X_2|X_1) = \frac{\beta}{\alpha+\beta}H(\alpha) + \frac{\alpha}{\alpha+\beta}H(\beta).$$

   (c) What values of α and β maximize the entropy rate?

2. Prove that entropy rate of random walk on a simple combinatorial graph (For details visit: `https://en.wikipedia.org/wiki/Graph_theory`) is
$$H(\mathcal{X}) = \log(2E) - H\left(\frac{E_1}{2E}, \frac{E_2}{2E}, \dots \frac{E_m}{2E}\right),$$
where E_i is number of edges emanating from node i and E is the total number of edges in the graph.

3. For any countably infinite set of codewords that form a prefix code, the codeword lengths satisfy $\sum_{i=1}^{\infty} D^{-l_i} \leq 1$. Conversely, given any $l_1, l_2, \dots$ satisfying the extended Kraft inequality, we can construct a prefix code with these codeword lengths. Notations have their conventional meaning.

4. Let the random variable X have five possible outcomes $\{1, 2, 3, 4, 5\}$. Consider two distributions $p(x)$ and $q(x)$ on this random variable mentioned in the table 1.

   (a) Calculate $H(p)$, $H(q)$, $D(p||q)$, and $D(q||p)$.

   (b) The last two columns represent codes for the random variable. Verify that the average length of C_1 under p is equal to the entropy $H(p)$. Thus, C_1 is optimal for p. Verify that C_2 is optimal for q.

Table 1: Table for question 8

Symbols	$p(x)$	$q(x)$	$C_1(x)$	$C_2(x)$
1	$\frac{1}{2}$	$\frac{1}{2}$	0	0
2	$\frac{1}{4}$	$\frac{1}{8}$	10	100
3	$\frac{1}{8}$	$\frac{1}{8}$	110	101
4	$\frac{1}{16}$	$\frac{1}{8}$	1110	110
5	$\frac{1}{16}$	$\frac{1}{8}$	1111	111

Table 2: Table

		U_n		
		S_1	S_2	S_3
	S_1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
U_{n-1}	S_2	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
	S_3	0	$\frac{1}{2}$	$\frac{1}{2}$

(c) Now assume that we use code C_2 when the distribution is p. What is the average length of the codewords. By how much does it exceed the entropy p?

(d) What is the loss if we use code C_1 when the distribution is q?

5. Consider the three-state Markov process $U_1, U_2, \ldots$ having transition matrix given in the table 2. Design three codes C_1, C_2, C_3 (one for each state $1, 2$ and 3, each code mapping elements of the set of S_is into sequences of 0s and 1s, such that this Markov process can be sent with maximal compression by the following scheme:

   (a) Note the present symbol $X_n = i$.

   (b) Select code C_i.

   (c) Note the next symbol $X_{n+1} = j$ and send the codeword in C_i corresponding to j.

   (d) Repeat for the next symbol. What is the average message length of the next symbol conditioned on the previous state $X_n = i$ using this coding scheme? What is the unconditional average number of bits per source symbol? Relate this to the entropy rate $H(U)$ of the Markov chain.

6. Which of the following codes are

   (a) Uniquely decodable?

   (b) Instantaneous?

$$
\begin{aligned}
C_1 &= \{00, 01, 0\} \\
C_2 &= \{00, 01, 100, 101, 11\} \\
C_3 &= \{0, 10, 110, 1110, \ldots\} \\
C_4 &= \{0, 00, 000, 0000\}
\end{aligned}
$$