Indian Institute of Technology Kharagpur
Information and Coding Theory (MA41024/MA60020)
Assignment - 1, Spring Semester 2020-21

Last Date of submission: January 31, 2021

1. Let X and Y be random variables with alphabets $X = Y = \{1, 2, 3, 4, 5\}$ and joint distribution $P(X, Y)$ is given by

$$\frac{1}{25}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 \\ 0 & 3 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix}.$$

Determine $H(X), H(Y), H(X|Y), H(Y|X)$, and $I(X;Y)$.

2. What is the minimum value of $H(p_1, \ldots, p_n) = H(P)$ as P ranges over the set of n-dimensional probability vectors? Find all P that achieve this minimum.

3. Let $C_\alpha = \sum_{n=2}^{\infty} \frac{1}{n(\log(n))^\alpha}$. Prove that
   (a) $C_\alpha \leq \infty$ if $\alpha > 1$ and , $C_\alpha = \infty$ if $0 \leq \alpha \leq 1$.
   (b) $p_\alpha(n) = [C_\alpha n(\log(n))^\alpha]^{-1}$ for $n = 2, 3, \ldots \infty$ is a probability distribution.
   (c) $H(p_\alpha) < \infty$ for $\alpha > 2$ and $H(p_\alpha) = \infty$ for $1 < \alpha \leq 2$.

4. Prove that the function $\rho(X, Y) = H(X|Y) + H(Y|X)$ is a metric on the set of all random variables defined on same sample space. Prove that

$$\begin{aligned} \rho(X, Y) &= H(X) + H(Y) - 2I(X;Y) \\ &= H(X, Y) - I(X;Y) \\ &= 2H(X, Y) - H(X) - H(Y). \end{aligned}$$

5. Let X denote the random variable which measures the number of tosses required for a coin ultil the first tail apears. Then
   (a) Find the entropy of X if the coin is fair
   (b) Next assume the coin to be unfair with p being the probability of getting a tail. Find the entropy.

6. Let $P = (p_1, p_2, \ldots)$ be a pmf corresponding to a sample space space which is countably infinite. Then show that if

$$\sum_{n=1}^{\infty} p_n \log n$$

converges then $H(P)$ is finite. Is the converse true? Justify your example.

7. Let P, Q and R be probability distributions on $S = \{s_1, s_2, \ldots, s_n\}$, with $P(s_i) = p_i$, $Q(s_i) = q_i$ and $R(s_i) = r_i$, $1 \leq i \leq n$. Then show that $D(P||R) = D(P||Q) + D(Q||R)$ if and oly if

$$\sum_{i=1}^{n} (p_i - q_i) \log(q_i/r_i) = 0.$$

8. Show that a cascade of n identical binary symmetric channels (BSCs) each with error probability p is equivalent to a single BSC with error probability $\frac{1}{2}[1 - (1 - 2p)^n]$, and hence

$$\lim_{n \to \infty} I(X_0; X_n) = 0$$

if $p \notin \{0, 1\}$.

9. Suppose X, Y and Z are discrete random variables. Then show the following:

   (a) $I(X;Y|Z) = I(Y;X|Z)$

   (b) $I(X;Y,Z) = I(X;Z) + I(X;Y|Z)$

   (c) $I(X;Y|Z) \geq 0$ if and only if (X, Z, Y) is a Markov chain.

All the best!!