Algebraic Riccati Equation

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GIAN course on Singular Optimal Control, 2016

Outline

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3. Peter Lancaster and Leiba Rodman, Algebraic Riccati Equations, Oxford University Press, 1995.
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5. W. T. Reid, Riccati Differential Equations, in volume 86 of Mathematics in Science and Engineering, Academic Press, New York, 1972.
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About Riccati and his equation

$$C + XA + DX - XBX = 0$$

Special case: A, B, C, D are real/complex numbers!!

- Count Jacopo Riccati was born in Venice on May 28, 1676.
- His father died when he was only ten years old.
- In 1693, he enrolled at the University of Padua as a student of law.
- Stefano was fond of Isaac Newton's Philosophiae Naturalis Principia Mathematica, which he passed on to young Riccati around 1695.
- After graduating on June 7, 1696, he married Elisabetta dei Conti d'Onigo on October 15, 1696. She bore him 18 children, of whom 9 survived childhood.

- He did not follow any lecture courses in mathematics. Basically, the profound knowledge of the self-taught man was acquired by reading.
- Riccati had far-reaching interests, ranging from mathematics to poetry, from physics to religion, as witnessed by his works and his rich library.
- He also believed that the brain should be better exercised in a variety of fields. As he wrote “Since adolescence, the mind should be educated to treasure the most eminent of sciences and the finest of arts. I do not want to claim that every topic should be probed in detail. Following one’s own talent and inclination, one should select at least one topic, and study it in depth. In the others, one should follow the example of the bee which sucks a drop of nectar from each flower.."

- He turned down many notable invitations, including the most appealing one of becoming president of St. Petersburg's Academy (1725). He also refused the chair of Mathematics at the University of Padua and the invitation to the Court of Wien as Aulic Adviser.
- On April 2, 1754, he had a sudden bout of fever. A fortnight later, on April 15, he passed away.

- Riccati's interest evolved around scalar equations of the type
$$\dot{x} = ax^2 + bx + c$$
with time varying or constant parameters.
- Besides a number of further equations of first order
$$\dot{x} = \alpha t^p x^q + \beta t^m$$
where m, p, and q are constants.
- he was particularly attracted by the equation
$$\ddot{x} = \alpha t^m$$
which he called "misleading equation".
- A generalization of the equation into a matrix form (the matrix Riccati equation) plays a major role in many design problems of modern engineering, especially filtering and control.

NARE

$$C + XA + DX - XBX = 0$$

where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{m\times n}$, $D \in \mathbb{C}^{m\times m}$ are known.

Find $X \in \mathbb{C}^{m\times n}$.

Construct:

$$\mathcal{H} = \begin{bmatrix} A & -B \\ -C & -D \end{bmatrix} \in \mathbb{C}^{(m+n)\times(m+n)}$$

Observations

1. X satisfies the NARE if and only if

$$\mathcal{H}\begin{bmatrix} I_n \\ X \end{bmatrix} = \begin{bmatrix} I_n \\ X \end{bmatrix}(A - BX)$$

$$\Rightarrow \quad \mathcal{H}\mathcal{V} \subseteq \mathcal{V}$$

where

$$\mathcal{V} = \left\langle \begin{bmatrix} I_n \\ X \end{bmatrix} \right\rangle \rightarrow \text{Graph space}$$

2. $$\lambda \in \Lambda(A - BX) \Rightarrow \lambda \in \Lambda(\mathcal{H})$$

3. $$\Lambda(\mathcal{H}) = \Lambda(A - BX) \cup \Lambda(-D + XB)$$

Finding a solution of the NARE is equivalent to finding the basis of an invariant subspace of $\mathcal{H}$.

Observation:

1. If $\begin{bmatrix} Y \\ Z \end{bmatrix}$ spans an n-dimensional subspace $\mathcal{V}$ of $\mathcal{H}$ such that $Y \in \mathbb{C}^{n\times n}$ is invertible, then $X = ZY^{-1}$ is a solution of the NARE.
2. The solution X is related to the eigenvectors (Jordan chains) of $A - BX$.

The solution X is associated with a set of eigenvalues of $A - BX$ but it is NOT a one-one correspondence!!

$$A = \begin{bmatrix} 3 & 0 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}, D = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$$

has solution

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Indeed,

$$\mathcal{H} \begin{bmatrix} I_n \\ X \end{bmatrix} = \begin{bmatrix} I_n \\ X \end{bmatrix} U, \quad \mathcal{H} \begin{bmatrix} I_n \\ Y \end{bmatrix} = \begin{bmatrix} I_n \\ Y \end{bmatrix} V$$

where

$$U = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, V = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

have the same eigenvalues.

- In most applications, the required solutions are associated with a subset of n eigenvalues of $\mathcal{H}$ corresponding to a unique n-dimensional invariant subspace.
- In other situations, where there exist different invariant subspaces associated with the same set of eigenvalues, some additional properties are needed in order to select the subspace of interest.
- This choice is generally made relying on the spectral properties of the matrix $\mathcal{H}$.

Splitting properties

For any k-tuple of complex numbers, in particular the eigenvalues of a matrix.

- The tuple $\omega = (\lambda_1, \dots, \lambda_k)$ of k complex numbers has a (k_1, k_2) splitting with respect to the imaginary axis or c-splitting if $k = k_1 + k_2$, $k_i \geq 1, i = 1, 2$, and there exists a permutation π of $\{1, 2, \dots, k\}$ such that

$$
\begin{aligned}
\lambda_{\pi(i)} &\in \mathbb{C}_{\leq}, i = 1 : k_1 \\
\lambda_{\pi(i)} &\in \mathbb{C}_{\geq}, i = k_1 + 1, \dots, k.
\end{aligned}
$$

$$\mathbb{C}_{\leq} := \{z \in \mathbb{C} : \mathrm{re}(z) \leq 0\}, \ \mathbb{C}_{\geq} := \{z \in \mathbb{C} : \mathrm{re}(z) \geq 0\}$$

Let $\omega_1 = (\lambda_{\pi(1)}, \ldots, \lambda_{\pi(k_1)}),\ \omega_2 = (\lambda_{\pi(k_1+1)}, \ldots, \lambda_{\pi(k)})$.

- Proper splitting: A c-splitting is proper if ω_1 OR ω_2 does not intersect the imaginary axis.
- Strong splitting: A c-splitting is strong if both ω_1 AND ω_2 do not intersect the imaginary axis.
- Weak splitting: A c-splitting is weak if both ω_1 AND ω_2 intersect the imaginary axis.

Stability region

A matrix A is called c-stable if all the eigenvalues of A lie in the stability region

$$\mathbb{C}_< = \{z \in \mathbb{C} : \mathrm{re}(z) < 0\}.$$

An invariant subspace of a matrix is called stable if it is spanned by Jordan chains associated with the stable eigenvalues, similarly, a subspace is called anti-stable if it is spanned by Jordan chains associated with anti-stable eigenvalues.
An invariant subspace of a matrix is called weakly stable if it is spanned by Jordan chains associated with stable and critical eigenvalues, similarly, a subspace is called weakly antistable if it is spanned by Jordan chains associated with antistable and critical eigenvalues.

Theorem

If the eigenvalues of $\mathcal{H}$ have a proper (m, n) c-splitting and if $\mathcal{H}$ has an n-dimensional c-antistable graph invariant subspace, then the NARE has a unique solution X such that $\Lambda(A - BX) \subseteq \mathbb{C}_{>}$.

Fréchet derivative

The Fréchet derivative of a matrix function $\mathcal{F} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ at X is a linear function $\mathcal{L} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$, $E \mapsto \mathcal{L}(X, E)$ such that for all $E \in \mathbb{C}^{n\times n}$

$$\mathcal{F}(X + E) - \mathcal{F}(X) - \mathcal{L}(X, E) = o(\|E\|)$$

1. The Fréchet derivative, if it exists, is unique.
2. A function $\mathcal{F}$ is said to be Fréchet differentiable at X if there exists its Fréchet derivative at X
3. The notation $\mathcal{L}(X, E)$ can be read as “the Frechet derivative of $\mathcal{F}$ at X in the direction E", or “the Fréchet derivative of $\mathcal{F}$ at X applied to the matrix E".

Consider the Riccati operator $\mathcal{R} : \mathbb{C}^{m\times n} \to \mathbb{C}^{m\times n}$ defined by

$$\mathcal{R}(X) = C + XA + DX - XBX.$$

Then its Fréchet derivative is given by

$$\mathcal{L}(X, E) = EA + DE - EBX - XBE.$$

Consequently, $\mathcal{L}(X, E)$ can be represented in terms of matrix vector products as

$$vec(\mathcal{L}(X, E)) = \Delta_X vec(E)$$

where

$$\Delta_X = (A - BX)^T \otimes I_m + I_n \otimes (D - XB).$$

The matrix Δ_X coincides also with the Jacobian of $\mathcal{R}(X)$ with respect to the variables x_{ij} ordered columnwise.

The eigenvalues of

$$\Delta_X = (A - BX)^T \otimes I_m + I_n \otimes (D - XB)$$

are the sums of those of $A - BX$ and $D - XB$ (Homework!!).

A solution S of the NARE is called critical if the Jacobian Δ_S of the Riccati operator $\mathcal{R}(X)$ is singular at $X = S$.

Note that the matrix Δ_X can be singular only if $A - BX$ and $XB - D$ have a common eigenvalue. Since the eigenvalues of $\mathcal{H}$ are the union of the eigenvalues of $A - BX$ and of $XB - D$, therefore the spectral properties of $\mathcal{H}$ play a crucial role for the existence of critical solutions.

Shifting techniques

- Introduce a transform which convert $\mathcal{H}$ into a new matrix $\widehat{\mathcal{H}}$ such that the graph *c*-(anti)stable invariant subspace is the same for $\mathcal{H}$ and for $\widehat{\mathcal{H}}$.
- This transformation will be particularly useful when we have to compute a critical solution S. In fact, in this case the new (transformed) algebraic Riccati equation is such that S is still a solution, but the Jacobian of the new Riccati operator $\widehat{\mathcal{R}}(X)$ is not singular at $X = S$.

We describe the shift technique for NARE such that the corresponding matrix $\mathcal{H}$ is singular.

Let the eigenvalues $\lambda_i, i = 1, \ldots, m+n$, of $\mathcal{H}$ have the following (m, n) c-splitting

$$\mathrm{re}(\lambda_1) \leq \ldots \leq \mathrm{re}(\lambda_m) \leq 0 = \lambda_{m+1} \leq \mathrm{re}(\lambda_{m+2}) \leq \ldots \leq \mathrm{re}(\lambda_{m+n})$$

and thus the NARE has an (almost) c-antistabilizing solution S, such that $A - BS$ has eigenvalues $\lambda_{m+1} = 0, \lambda_{m+2}, \ldots, \lambda_{m+n}$. In particular,

$$\mathcal{H} \begin{bmatrix} I_n \\ S \end{bmatrix} = \begin{bmatrix} I_n \\ S \end{bmatrix} (A - BS).$$

The goal is to construct a new matrix $\widehat{\mathcal{H}}$ having the same graph invariant subspace of $\mathcal{H}$, spanned by

$$\begin{bmatrix} I_n \\ S \end{bmatrix}$$

where the matrix $A - BS$ is replaced by a matrix having eigenvalues $\eta, \lambda_{m+2}, \ldots, \lambda_{m+n}$, and $\eta \in \mathbb{C}_>$. In particular, the eigenvalue $\lambda_{m+1} = 0$ of $\mathcal{H}$ is shifted to the eigenvalue η of $\widehat{\mathcal{H}}$.

Theorem

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and let v be a nonzero vector such that $Av = \lambda_1 v$. For any nonzero vector x, the eigenvalues of $A + vx^*$ are $\lambda_1 + x^* v, \lambda_2, \ldots, \lambda_n$.

Construction of $\widehat{\mathcal{H}}$

1. Let $v \in \mathbb{C}^{m+n}$ be a nonzero vector such that $\mathcal{H}v = 0$
2. Let $p \in \mathbb{C}^{m+n}$ be such that $p^* v = 1$, and let η be a complex number with positive real part.
3. Define $\widehat{\mathcal{H}} = \mathcal{H} + \eta v p^*$

Then the eigenvalues of $\widehat{\mathcal{H}}$ are those of $\mathcal{H}$ except for the eigenvalue $\lambda_{m+1} = 0$ of $\mathcal{H}$ which is replaced by η (Homework!! use the previous Theorem).

Theorem

Let A be an $n \times n$ matrix and let v be an eigenvector of A, that is $Av = \lambda v$ for some λ. Let V be an $n \times m$ matrix whose columns span an invariant subspace of A of dimension m including v, so that $AV = VP$ for a suitable $P \in \mathbb{C}^{m \times m}$. Then, for any nonzero $x \in \mathbb{C}^n$, it holds that $(A + vx^*)V = V(P + \widetilde{v}\widetilde{x}^*)$ where $\widetilde{v}$ is the unique solution of $V\widetilde{v} = v$ and $\widetilde{x} = x^* V$. Moreover $P\widetilde{v} = \lambda\widetilde{v}$.

Thus

$$\widehat{\mathcal{H}} \begin{bmatrix} I_n \\ S \end{bmatrix} = \begin{bmatrix} I_n \\ S \end{bmatrix} R$$

where R is a rank-one modification of $A - BS$, having eigenvalues $\eta, \lambda_{m+2}, \ldots, \lambda_{m+n}$.

We may partition p, v, and $\widehat{\mathcal{H}}$ according to the partitioning of $\mathcal{H}$ as

$$p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \widehat{\mathcal{H}} = \begin{bmatrix} \widetilde{A} & -\widetilde{B} \\ -\widetilde{C} & -\widetilde{D} \end{bmatrix}$$

and hence

$$\widetilde{A} = A + \eta v_1 p_1^* \quad \widetilde{B} = B - \eta v_1 p_2^*$$
$$\widetilde{C} = C - \eta v_2 p_1^* \quad \widetilde{D} = D - \eta v_2 p_2^*.$$

Finally we have the new NARE

$$\widetilde{C} + X\widetilde{A} + \widetilde{D}X - X\widetilde{B}X = 0$$

which has the same solution X as the original NARE, where the eigenvalue $\lambda_{m+1} = 0$ of $A - BX$ is shifted to the eigenvalue η of $\widetilde{A} - \widetilde{B}X$.

Remark

1. if $\mathrm{re}(\lambda_{m+2}) > 0$, then X is the *c*-antistabilizing solution of the new NARE.
2. In the case where $\lambda_m = \lambda_{m+1} = 0$, so that X is a critical solution, if $\mathrm{re}(\lambda_{m+2}) > 0$, then we may show that X is a noncritical solution of the new NARE

$$C + XA + A^*X - XBX = 0$$

where $A, B, C \in \mathbb{C}^{n\times n}$, $B^* = B$, $C^* = C$.
Construct

$$\mathcal{H} = \begin{bmatrix} A & -B \\ -C & -A^* \end{bmatrix}$$

Observations

1. $\mathcal{H}$ is a Hamilton matrix: $\mathcal{J}\mathcal{H} = -\mathcal{H}^*\mathcal{J}$ where $\mathcal{J} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$.
2. The spectrum of $\mathcal{H}$ is symmetric with respect to the imaginary axis, i.e., the nonimaginary eigenvalues of $\mathcal{H}$ come in pairs $(\lambda, -\overline{\lambda})$,

The eigenvalues of $\mathcal{H}$ can be ordered in such a way that

$$\begin{aligned} \text{re}(\lambda_1) \leq \text{re}(\lambda_2) \leq \ldots \leq \text{re}(\lambda_n) \leq 0 \\ \leq \text{re}(\lambda_{n+1}) \leq \text{re}(\lambda_{n+2}) \leq \ldots \leq \text{re}(\lambda_{2n}). \end{aligned}$$

Thus the eigenvalues of $\mathcal{H}$ have an (n, n) c-splitting.

1. If $\mathcal{H}$ has no eigenvalues on the imaginary axis, then the splitting is strong and therefore there are unique, c-stable, and c-antistable invariant subspaces corresponding to the n eigenvalues with negative real part and positive real part, respectively. (Homework!!)
2. If the splitting is weak, then there can be more than one c-stable n-dimensional invariant subspace. However, if all the imaginary eigenvalues have even partial multiplicities, then there exists a unique, canonical weakly c-stable invariant subspace. (Homework!!)

Consider the scalar equation

$$\alpha x^2 + \beta x + \gamma = 0$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $\alpha \neq 0$. Then

$$\mathcal{H} = \begin{bmatrix} \frac{\beta}{2} & \alpha \\ -\gamma & -\frac{\beta}{2} \end{bmatrix}.$$

The eigenvalues of $\mathcal{H}$ are $\pm\frac{1}{2}\sqrt{\beta^2 - 4\alpha\gamma}$.

1. If $\sqrt{\beta^2 - 4\alpha\gamma} \neq 0$ then there are two 1-dimensional $\mathcal{H}$-invariant subspaces of $\mathbb{C}^2$, namely $\left\langle \begin{bmatrix} 1 \\ \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \end{bmatrix} \right\rangle$
2. If $\sqrt{\beta^2 - 4\alpha\gamma} = 0$ then there exists only one such subspace $S = \left\langle \begin{bmatrix} 1 \\ -\frac{\beta}{2\alpha} \end{bmatrix} \right\rangle$ which corresponds to the unique solution $x = -\beta/2\alpha$.

Recall the CARE

$$C + XA + A^*X - XBX = 0.$$

Note that X is Hermitian if and only if

$$\begin{bmatrix} I_n & X^* \end{bmatrix} \mathcal{J} \begin{bmatrix} I_n \\ X \end{bmatrix} = 0.$$

$\mathcal{J}$-Neutral

A subspace S is called $\mathcal{J}$-Neutral if $x^*\mathcal{J}y = 0$ for all $x, y \in S$.

Then it follows that if $S = \left\langle \begin{bmatrix} I_n \\ X \end{bmatrix} \right\rangle$ is an invariant subspace associated with the solution X, then X is Hermitian if and only if S is $\mathcal{J}$-neutral subspace of $\mathcal{H}$.

Theorem

If $\mathcal{H}$ has no purely imaginary eigenvalues and there exists a solution X such that $A - BX$ is stable, then X is the unique stabilizing solution, in particular X is Hermitian. If the purely imaginary eigenvalues of $\mathcal{H}$ correspond to even-sized Jordan blocks in the Jordan normal form of $\mathcal{H}$, and if there exists a solution X associated with the canonical c-stable invariant subspace, then X is Hermitian.

Proof: The columns of $\begin{bmatrix} I_n \\ X \end{bmatrix}$ span the canonical c-stable invariant subspace of $\mathcal{H}$. Since the equation is symmetric, the matrix X^* is a solution of the CARE as well, and

$$\mathcal{H} \begin{bmatrix} I_n \\ X^* \end{bmatrix} = \begin{bmatrix} I_n \\ X^* \end{bmatrix} (A - BX^*).$$

We show that the columns of $\begin{bmatrix} I_n \\ X^* \end{bmatrix}$ span the canonical *c*-stable invariant subspace of $\mathcal{H}$, and from the uniqueness of such a subspace we deduce that $X = X^*$.

Since $\begin{bmatrix} -X & I_n \end{bmatrix} \begin{bmatrix} I_n \\ X \end{bmatrix} = 0$, the columns of $\begin{bmatrix} -X^* \\ I_n \end{bmatrix}$ span the space orthogonal to the space spanned by $\begin{bmatrix} I_n \\ X \end{bmatrix}$. On the other hand, since left and right invariant subspaces corresponding to disjoint sets of eigenvalues are orthogonal (Homework!!), the left canonical *c*-antistable invariant subspace is orthogonal to the span of $\begin{bmatrix} I_n \\ X \end{bmatrix}$.

Thus, since $\begin{bmatrix} I_n \\ X \end{bmatrix}$ and $\begin{bmatrix} -X^* \\ I_n \end{bmatrix}$ span $\mathbb{C}^{2n}$, then we deduce that the left canonical c-antistable invariant subspace of $\mathcal{H}$ is spanned by $\begin{bmatrix} -X^* \\ I_n \end{bmatrix}$.

Moreover, $\begin{bmatrix} -X & I_n \end{bmatrix} \mathcal{H} = (XB - A^*) \begin{bmatrix} -X & I_n \end{bmatrix}$ implies that the matrix $XB - A^*$ collects the eigenvalues of $\mathcal{H}$ with positive real part and the imaginary eigenvalues of $\mathcal{H}$ with half partial multiplicity.

Thus $A - BX^* = -(XB - A^*)^*$ has the same Jordan structure as $A - BX$, and by the uniqueness of the canonical invariant subspace we have $X = X^*$.

Theorem

Let $B \succeq 0$.

1. There exists a unique Hermitian solution X_+ of the CARE such that the eigenvalues of $A - BX_+$ have nonpositive real part if and only if the pair (A, B) is c-stabilizable and the partial multiplicities of the pure imaginary eigenvalues of $\mathcal{H}$ (if any) are all even.
2. There exists a unique Hermitian solution X_- of the CARE such that the eigenvalues of $A - BX_-$ have nonnegative real part if and only if the pair $(-A, B)$ is c-stabilizable and the partial multiplicities of the pure imaginary eigenvalues of $\mathcal{H}$ (if any) are all even.

The Fréchet derivative of the Riccati operator (corresponding to CARE) takes the form

$$\mathcal{L}(X, E) = EA + A^*E - EBX - XBE$$

which is expressed in matrix form

$$vec(\mathcal{L}(X, E)) = \Delta_X vec(E)$$

where

$$\Delta_X = (A - BX)^T \otimes I_n + I_n \otimes (A^* - XB).$$

As is the case of NARE, a solution S of the CARE is called critical if the Jacobian Δ_S is singular at $X = S$.

Observations

1. The matrix $\mathcal{H}$ is Hamiltonian and its eigenvalues come in pairs $(\lambda, -\overline{\lambda})$. From the properties of the eigenvalues of Kronecker products, if $\mathcal{H}$ has no pure imaginary eigenvalue, then any *c*-(anti)stabilizing solution of the CARE is not critical.
2. If $\mathcal{H}$ has some eigenvalues on the imaginary axis, that their partial multiplicities are even, and that there exists a solution S of the CARE associated with the canonical weakly *c*-stable invariant subspace. From the Theorem proved earlier, the matrix S is the unique (almost) *c*-stabilizing Hermitian solution S. This solution is critical. In fact, S is such that $A - BS$ has at least one purely imaginary eigenvalue λ and the matrix $(A - BS)^* = A^* - SB$ has the eigenvalue $-\lambda$. Therefore the Jacobian Δ_S is singular.

Shifting the CARE

We describe the shift technique for the CARE with a singular Hamiltonian $\mathcal{H}$ without changing the Hamiltonian structure of the new $\widehat{\mathcal{H}}$.

Assumptions

1. Assume that $\mathcal{H}$ has a zero eigenvalue with partial multiplicity 2 and that there are no pure imaginary eigenvalues.
2. Assume also that the CARE has an (almost) *c*-stabilizing solution S, such that the eigenvalues of $A - BS$ are $\lambda_1, \ldots, \lambda_{n-1}, 0$. The solution S is critical, since zero is a double eigenvalue of $\mathcal{H}$.

In order to maintain the Hamiltonian property we need to apply a double shift by moving the zero eigenvalues to a pair of eigenvalues η and $-\eta$, where $\eta \in \mathbb{R}, \eta \neq 0$. Here and hereafter we assume that $\eta < 0$.

Procedure

1. Denote by $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ a right eigenvector of $\mathcal{H}$ corresponding to the zero eigenvalue.
2. Then $w^* = \begin{bmatrix} v_2^* & -v_1^* \end{bmatrix}$ is a left eigenvector of $\mathcal{H}$ corresponding to the zero eigenvalue.
3. Since $\lambda_n = 0$ is an eigenvalue of $\mathcal{H}$ with partial multiplicity 2, $w^* v = 0$. (Homework!!)

Then it follows that the double shift which maps one zero eigenvalue in $\eta \neq 0$ and the other zero eigenvalue in $-\eta$ is given by

$$\widehat{\mathcal{H}} = \mathcal{H} + \eta \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} p_1^* & p_2^* \end{bmatrix} - \eta \begin{bmatrix} q_1^* \\ q_2^* \end{bmatrix} \begin{bmatrix} v_2^* & -v_1^* \end{bmatrix}$$

where the vectors p_1, p_2, q_1, q_2 are chosen in such a way that

$$\begin{aligned} p_1^* v_1 + p_2^* v_2 &= 1 \\ v_2^* q_1 - v_1^* q_2 &= 1. \end{aligned}$$

Observe

1. $\widehat{\mathcal{H}} v = \eta v$ and $w^* \widehat{\mathcal{H}} = -\eta w^*$
2. $\widehat{\mathcal{H}} \begin{bmatrix} I_n \\ S \end{bmatrix} = \begin{bmatrix} I_n \\ S \end{bmatrix} \widehat{R}$ (Homework!!, use $w^* v = 0$)

Finally, for the Hamiltonian property of $\widehat{\mathcal{H}}$ we must have

$$\mathcal{J}^T \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} p_1^* & p_2^* \end{bmatrix} - \begin{bmatrix} q_1^* \\ q_2^* \end{bmatrix} \begin{bmatrix} v_2^* & -v_1^* \end{bmatrix} \right) \mathcal{J}$$

$$= \quad - \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} p_1^* & p_2^* \end{bmatrix} - \begin{bmatrix} q_1^* \\ q_2^* \end{bmatrix} \begin{bmatrix} v_2^* & -v_1^* \end{bmatrix} \right)$$

This yields (after some algebraic manipulation, Homework!!)

$$p_1 = v_2 + \theta v_1, p_2 = \theta v_2 - v_1, q_1 = \theta v_2, q_2 = -\theta v_1, \theta = \|v\|_2^2.$$

In this way, $\widehat{\mathcal{H}}$ is still Hamiltonian, and the matrix coefficients are given by

$$\begin{aligned} \widehat{A} &:= A + \eta(v_1 v_2^* + \theta v_1 v_1^* - \theta v_2 v_2^*) \\ \widehat{B} &:= B - \eta(\theta v_1 v_2^* + \theta v_2 v_1^* - v_1 v_1^*) \\ \widehat{C} &:= C - \eta(\theta v_2 v_1^* + \theta v_1 v_2^* + v_2 v_2^*). \end{aligned}$$

If $v_2^* v_1 \neq 0$, a simpler formula can be obtained as follows

$$p_1 = \theta v_2, q_1 = \theta v_1, p_2 = q_2 = 0, \theta = 1/(v_2^* v_1)$$

and in that case

$$\widehat{A} := A, \widehat{B} := B - \eta v_1 v_1^*, \widehat{C} := C - \eta v_2 v_2^*$$

form $\widehat{\mathcal{H}}$ a Hamiltonian matrix.

The eigenvalues of $\widehat{\mathcal{H}}$ are $\lambda_i = -\lambda_{n+i}, i = 1, \ldots, n-1$, and $\lambda_n = -\lambda_{n+1} = \eta$. Therefore, if $\eta < 0$, the almost *c*-stabilizing solution *S* of the original CARE is now the *c*-stabilizing solution of the new CARE. Moreover, while the solution S is critical for the original CARE, the same solution *S* is not critical for the new CARE.

Example

Consider the CARE

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

and $X_+ = \begin{bmatrix} 0 & 1 \\ 1 & 1/2 \end{bmatrix}$ is the unique Hermitian solution. Moreover, it is a critical solution since $A - BX_+ = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}$. The eigenvalues of the associated matrix

$$\mathcal{H} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix}$$

are $-1, 0, 1$, where 0 has partial multiplicity 2. In particular, $\mathcal{H}v = 0$

where

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Applying the double shift technique with $\eta = -2$ we obtain

$$\widehat{A} = \begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix}, \widehat{B} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, \widehat{C} = \begin{bmatrix} 0 & 2 \\ 2 & 4 \end{bmatrix}.$$

The eigenvalues of the matrix $\widehat{\mathcal{H}}$ are $-2, -1, 1$, and 2, and the matrix X_+ is the unique *c*-stabilizing solution of the CARE with matrix coefficients $\widehat{A}$, $\widehat{B}$, $\widehat{C}$ moreover X_+ is noncritical.

Invariant subspace method

The most straightforward way to find an invariant subspace is through eigenvectors, but the procedure may lead to unexpected numerical problems since it may happen that the invariant subspace to be computed is well conditioned, while some single eigenvector is not. A more numerically sound procedure is based on the Schur decomposition.

Assumption

Assume that the CARE has a unique *c*-stabilizing solution X_+, in particular, the eigenvalues of the matrix $\mathcal{H}$ of have a strong (n, n) *c*-splitting since the matrix $\mathcal{H}$ is Hamiltonian and the CARE has a *c*-stabilizing solution.

Thus

$$\mathcal{H}\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} \Gamma$$

where Γ is an $n \times n$ c-stable matrix, then

$$X_+ = UV^{-1}.$$

Therefore, in order to compute the c-stabilizing solution X_+ it is sufficient to compute a basis for the c-stable invariant subspace of $\mathcal{H}$, which is unique by the splitting of the eigenvalues of $\mathcal{H}$.

This task can be performed efficiently by using the semiordered real Schur decomposition of $\mathcal{H}$. More specifically, the matrix $\mathcal{H}$ is factored as $\mathcal{H} = QRQ^T$, where Q and R are partitioned into four $n \times n$ blocks

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \quad R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix},$$

and

- Q is orthogonal
- R_{11} and R_{22} are block upper triangular matrices with diagonal blocks of size at most 2, moreover, the matrix R_{11} is c-stable, that is, R_{11} collects the eigenvalues of $\mathcal{H}$ with negative real part.

The first n columns of Q span the c-stable invariant subspace of $\mathcal{H}$ so that

$$X_+ = Q_{21} Q_{11}^{-1}$$

is the stabilizing solution

The semiordered real Schur decomposition can be computed by the MATLAB functions schur and ordschur according to the following two steps:

1. `[U,T] = schur(H,'real')` computes a real Schur decomposition of $\mathcal{H}$ by means of the *QR* algorithm
2. `[Q,R] = ordschur(U,T,select)` swaps the diagonal blocks by means of orthogonal transformations in such a way that the eigenvalues with indices selected by the logical vector select are the eigenvalues of the leading $n \times n$ submatrix of R.

The same approach can be applied in more general contexts, say, in the case of a NARE. In this generalized version, one has to identify which eigenvalues of the associated matrix $\mathcal{H}$ correspond to the sought solution. Then, by a sequence of unitary transformations, one must put the desired eigenvalues in the upper left block of an ordered Schur form of $\mathcal{H}$.

Standard Newton's method

Newton's method is the customary numerical tool for solving scalar nonlinear equations. Given an equation $f(x) = 0$, where f is continuously differentiable in a neighborhood of a solution $\alpha \in \mathbb{C}$, Newton's method generates a sequence $\{x_k\}$ defined by the recurrence

$$x_{k+1} = x_k - f(x_k)/f'(x_k)$$

for a suitable initial guess x_0 and whose limit is α.
The method can be used also for solving equations of the kind $F(X) = 0$, where $F : V \to V$ is a differentiable operator in a Banach space (we are interested only in the case in which V is $\mathbb{C}^{m \times n}$). The sequence is defined by

$$X_{k+1} = X_k - (F'_{X_k})^{-1} F(X_k), X_0 \in V,$$

where $F'(X)$ is the Fréchet derivative of F at the point X.