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On the Pareto optimality of variance reduction simulation techniques in structural reliability

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ABSTRACT

Estimation of low failure probabilities in high dimensional structural reliability problems generally involves a trade-off between computational effort and accuracy of the estimate, whether efficient sampling techniques have been employed or not. While a substantial effort continues to be made by the community to develop and benchmark new and efficient sampling schemes, the limits of performance of a given algorithm, e.g., what is the best attainable accuracy of the method for a fixed computational effort and if that is good enough, have not received comparable attention. However, such insights could prove valuable in making the right choice in solving a computationally demanding reliability problem. In a multi-objective stochastic optimization formulation, these questions yield the so-called Pareto front or the set of non-dominated solutions: solutions that cannot be further improved without worsening at least one objective. Posteriori user defined preferences can then be applied to rank members of the Pareto set and obtain the best strategy. We take up two classes of variance-reducing algorithms – importance sampling (IS) and subset simulations (SS) – and apply them to a range of benchmarked reliability problems of various size and complexity to bring out the issue of optimality and trade-off between accuracy and effort. The design variables are variously of categorical, discrete as well as continuous types and the stochastic multi-objective optimization without recourse is solved using Genetic Algorithms. In each case, we ascertain the best possible accuracies that a given method can achieve and identify the corresponding design variables. We find that the proposal pdf does have an effect on the efficiency of SS, the FORM design point is not always the best sampling location in IS and setting the sensitivity parameter associated with Adaptive Importance Sampling at 0.5 does not guarantee optimal performance. In addition to this the benefits of using SS for high dimensional problems are reinforced. We also show that the Pareto fronts corresponding to different methods can intersect indicating that more is not always better and different solution techniques for the same problem may be required in different computational regimes.

1. Introduction

For a distributed structure with several potential critical locations and failure modes (such as shear and flexure), subject to time dependent loads and possessing time- and space-dependent material properties, the reliability function estimates the probability that the capacity, C, exceeds the demand, D, in each failure mode, at all locations and at all times that the structure is in service:

$$\mathrm{Rel}(t) = 1 - P_f(t) = P[C_j(\underline{x}, \tau) \geqslant D_j(\underline{x}, \tau), j \leqslant J, \forall \tau \in (0, t), \forall \underline{x} \in \Omega] \quad (1)$$

where Ω is the set of critical locations of the structure, J is the total number of failure modes at each critical location, and t is total time horizon. Both capacity and demand of the structure are generally functions of space and time and constitute a multidimensional stochastic process.

The structural reliability problem in its most general formulation is thus infinite dimensional both in time and space which of course makes it computationally intractable; hence various levels of simplification are adopted. If there is only one critical location with only one failure mode, and demand and capacity are time invariant as well, we have the most basic formulation: a time-independent element level reliability problem which typically is described by a few basic variables and can be solved by elegant geometric techniques such as FORM and SORM. Monte Carlo simulation based techniques can also be adopted with ease. Time invariant problems with more than one failure mode and/or location can be modeled as a system reliability problem with appropriate unions and intersection of element level limit states

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and can still be tackled with FORM/SORM although with increasing approximation. For such problems, simulation based techniques, especially with some efficient sampling scheme, may appear more desirable.

When the time dependent nature of C and/or D of the critical element cannot be neglected, the next level of complexity in reliability problems involves condensing the time dimension to a finite number of discrete points by modeling the load as a stationary pulse process and the capacity as a non-random function of time. A higher level of complexity occurs when stationarity can no longer be assumed due either to non-stationary excitation or to stochastic degradation: the first passage into failure by the process C–D at the critical location may need to be solved by direct simulations in the time domain.

In addition to modeling temporal randomness, spatial randomness may need to taken into account for distributed problems. Random fields describing the spatially varying quantities are discretized according to the set of critical locations and the discretization scheme adopted to solve the problem as using a finite element formulation, and can involve local averaging, series approximations, interpolation etc. [1]. Spatial randomness thus increases the dimensionality of the reliability problem; it also affects the statistical dependence between safety margins at different locations both at the same instant and at different instants of time.

For most real life structures one thus finds a high dimensional reliability problem [2]. In addition to this, very low failure probabilities are typically associated with structures owing to the high level of safety that society has come to expect of structures. A brief discussion on acceptable failure probabilities of different types of structures can be found in Bhattacharya et al. [3].

When the number of random variables become large, an important issue is how much one can trust the reliability index obtained from analytical approximate methods like FORM. In general the optimization procedure associated with FORM becomes unmanageable in high dimensions and it is advisable that simulation techniques be used [4]. The computation time increases with the number of random variables and if gradient computations are done numerically, the number of limit state function calls is proportional to the number of random variables [5]. Adhikari [6] estimated the failure probability using asymptotic distributions and derived a modified beta (actual beta estimate does not give correct values) for such cases. Schueller et al. [7] suggested that for dimensions greater than 30 FORM yields inaccurate solutions.

Simulation based approaches to structural reliability computation offer a far greater flexibility and can address many of the shortcomings of analytical based methods. At the same time, simulations have their limitations in terms of speed, size and accuracy. After all, simulation based algorithms are basically numerical statistical sampling schemes, and can never be free of sampling errors. All pseudo random number generators (as opposed to true random bit generators that are accurate but very slow [8]) suffer from finite periods (although the Mersenne Twister algorithm has one of the longest periods [9]). The gap therefore, at any given point of time, between computational need and computational resource, i.e., between the grand problem that the community would like to solve and the problem it is able to tackle due to hardware and/or algorithmic limitations, has always existed. Naturally, then, continual efforts have been made by the community to invent clever and efficient simulation schemes [7,10–15].

While substantial effort continues to be made to develop and benchmark new and efficient sampling schemes, the limits of performance of a given algorithm, e.g., what is the best attainable accuracy of the method for a fixed computational effort and if that is good enough, have not received comparable attention. We believe that such inquiries could prove valuable in making the right choice in solving a computationally demanding reliability problem. We investigate, from a multi-objective stochastic optimization viewpoint, two classes of variance-reducing algorithms – importance sampling and subset simulations – in order to bring out the issue of optimality and trade-off between accuracy and effort.

Even though comparative studies have been undertaken in the past, no author has tried to formulate it as a multi-objective optimization problem to the best of our knowledge. Previous comparative studies have only looked at superiority of one method over the other [7,12,15]. In addition to a comparative study the present work gives us an idea about the optimal combination of parameters to be used for a given levels of computational resources. Posteriori preferences of the user can be used to select the optimum method along with the optimum design parameters for best performance but is outside the scope of this paper.

We determine for a set of benchmark problems in structural reliability the best accuracy that a method can achieve given a fixed computational effort. We find that some conventional wisdoms, such as IS should be centered on the FORM design point and SS is not affected by the choice of proposal pdf, may not have much merit. We also demonstrate that the Pareto fronts corresponding to different methods can intersect indicating that more is not always better and different solution techniques for the same problem may be required at different regimes.

The structure of the paper is as follows. The next two sections give a brief overview of variance reduction techniques in structural reliability and multi-objective stochastic optimization problems in engineering. We then demonstrate the concepts of design variables, random objectives, solutions of various ranks, and the Pareto front through a simple one dimensional reliability problem. Following this, six benchmark reliability problems, in order of increasing complexity, are taken up in detail.

2. Variance reduction techniques in estimating reliability

If the reliability problem given in Eq. (1) can be expressed in terms of a finite number of basic variables, $\mathbf{X}$, whose membership in the failure region F can be verified by a finite number of binary checks summarized by the indicator function I_F, the failure probability is:

$$P_f = P(\mathbf{X} \in F) = \int I_F(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} = E[I_F(\mathbf{X})] \tag{2}$$

There are a number of ways of solving the above integral as we have discussed above. The most robust simulation based technique for estimating P_f in Eq. (2) is the basic Monte Carlo Simulation (MCS) [16]. However, while MCS gives an unbiased estimate of P_f, the coefficient of variation (c.o.v.) of the MCS estimator is $\sqrt{(1-P_f)/(P_f N)}$ which clearly shows the unfavourable relation between accuracy and effort involved in basic MCS.

2.1. Importance sampling and its variants

In order to overcome the problem of low efficiency associated with basic Monte Carlo techniques, a number of variance reduction techniques have been proposed over the years [17,18], the most widely used being Importance Sampling (IS) [16] whose basic idea is to carry out the simulations in a region which is considerably closer to the limit state:

$$P_f = P(X \in F) = \int I_F(x)[f_X(x)/h_\nu(x)]h_\nu(x)dx = E\left[I_F(\nu)\frac{f_X(\nu)}{h_\nu(\nu)}\right] \tag{3}$$

where $h_\nu(.)$ is the importance sampling density function which one would ideally like to centre on the point of maximum likelihood. If $h_\nu(.)$ is suitably chosen, one may generate a relatively large number of samples in the failure region and hence estimate low probabilities of failure with relatively small effort. Theoretically, the variance of the estimated failure probability can be reduced to 0 if the IS density is chosen appropriately, which however, requires the knowledge of the true probability of failure for the given problem and is not practicable. Hence the major issue involving this method is the choice of the IS density.

In Adaptive Importance Sampling (AIS) the selection of sampling density is made adaptive [19]. A rough estimate of the failure probability (found using basic MCS) is used to construct a kernel density. However, when dealing with very low failure probabilities finding an initial estimate of the failure probability using Monte Carlo methods becomes inefficient. Au and Beck [20] proposed an improved Adaptive Importance Sampling technique (henceforth denoted as AIS:MCMC) where they used Markov chains (generated using the Metropolis Hastings algorithm) to populate the variable space and get the kernel sampling density which asymptotically yields the optimal sampling density. For both these adaptive methods the number of samples used to construct the kernel density, the number of samples generated from the kernel sampling density and the choice of the sensitivity parameter for window width factor computation (window width factors are used to construct the kernel sampling density) are the important design parameters. For the AIS:MCMC variant the standard deviation of the proposal probability density function for generation of Markov chains becomes crucial too. In addition to this the initial sample chosen (which must lie in the failure region) for MCMC simulation can affect the algorithm's efficiency.

Importance sampling is best used when the limit state does not yield multiple points of maximum likelihood. The implementation becomes quite complicated if multiple limit states are involved as in cases of dynamic systems. Such cases have been solved using partial sampling at different points of maximum likelihood and then adding the total number of failures obtained from each such partial sampling. This naturally removes the simplicity associated with basic MCS by introducing several analytical steps. At the same time, versatility and accuracy are compromised. For applying IS to problems in high dimensions one has to ensure that the problem is not degenerate. The conditions under which IS can be applied in high dimensions have been discussed by Au and Beck [21]. The solution provided by the authors is essentially taking the IS estimate nearer and nearer to the basic MCS estimate (which again takes one back to the issue of inefficiency). To overcome the issues pertaining to IS Au and Beck [22] proposed Subset Simulation to solve reliability problems in high dimensions and having low failure probability which is discussed next.

2.2. Subset simulation and its variants

The basic principle in subset simulation is the expression of the probability of any given event as a product of a number of conditional probabilities of nested events [22–26]:

$$P(X_n) = P(X_1)P(X_2|X_1)\dots P(X_n|X_{n-1}) \quad (4)$$

where, $\{X_1, X_2, \dots, X_n\}$ is a given set of nested events and X_i is necessarily a subset of X_j if $i < j$. Values of $P(X_1)$, $P(X_2|X_1)$ etc. are typically large compared to $P(X_n)$, and their product accurately yields the desired probability. If we now define X_i as the event of exceeding the ith limit state, then the reliability problem gets formulated as one of evaluating $(n-1)$ conditional limit state probabilities. The product of these n probabilities, each of which can be designed to be adequately high by an appropriate selection of the intermediate limit states, gives us the probability of exceeding the original limit state.

The probability of exceeding the first limit state, $P(F_1)$, can be directly computed using basic MCS. For populating the intermediate failure levels Markov Chain Monte Carlo simulation (also referred to as MCMC "moves" in the following) is used. A modified Metropolis Hastings algorithm is used for the MCMC simulation. The main design issues of subset simulation are the spread of the proposal pdf, the conditional probability and the number of samples generated at an intermediate failure level. The proposal pdf of the modified MH algorithm decides the transition probability matrix of the Markov chain that is generated. If the variance is too low then the failure space will not be spanned adequately, whereas if the variance is too high a large number of samples will fall into the rejection space. If there are too many rejections the Markov chain produced will not span the variable space properly. Both conditions may lead to a situation where the failure state is never reached. Hence the variance of the proposal pdf should be judiciously chosen as it decides the efficiency of the algorithm. Recently Zuev and Katafygiotis [27] have proposed a modification for the Metropolis Hastings algorithm to reduce the rejections in the algorithm. However this comes at the cost of a higher computational effort.

For ease of implementation the conditional probability, $p_c = P[F_{i+1}|F_i], (i < n-1)$, is selected *a priori*. The intermediate failure levels are then set adaptively depending on the problem. This conditional probability that is selected also affects the efficiency of the algorithm. A low p_c will require a lower number of intermediate failure levels, however the number of samples required at each intermediate level will go up. Higher number of samples at each intermediate failure level clearly will increase the accuracy of the results at a disproportionately higher computational effort. Samples generated using the MH algorithm are not independent of each other and is partly the reason for the high c.o.v. that is typically observed in Subset Simulation with MCMC. The number of samples generated from each seed may not be enough to ensure burn-in so that the target distribution is reached. A discussion on the issues related to the modified Metropolis Hastings algorithm used by Au and Beck can be found in Katafygiotis and Zuev [28].

There are two variants of Subset Simulation that are useful when dealing with dynamics problems. They are Subset Simulation with Splitting [23] and Hybrid Subset Simulation [24]. Both these methods apply the concept of splitting of stochastic processes for computation of conditional failure probabilities. This helps in avoiding repeated generation of the whole time history of the system. It has been observed that Hybrid SS performs better than both SS:MCMC and SS Splitting for dynamics problems [25]. However SS Splitting outperforms SS:MCMC only for certain specific problems [25]. The present study is limited to SS:MCMC as it is the most general and the oldest among the above mentioned variants.

3. Tradeoff between computational effort and accuracy

For structural reliability computations, one would want to minimize computational effort as well as the error in the failure probability estimate. Of course these two objectives are conflicting in nature i.e., low error requires high computational effort and vice versa. Further, effort and error are random in nature due to the very presence of random sampling at the core of any Monte Carlo simulation based algorithm. The idea of a trade-off is clear here and the problem at hand can be formalized as a Multi-Objective Stochastic Optimization (MOSO) problem with two objectives – expected computational effort and expected error in estimate. Due to random sampling involved in each of the algorithms the objectives themselves will be random in nature. To account for this randomness a multi-objective stochastic optimization framework is necessary.

We adopt the following formulation of the multi-objective stochastic optimization problem without recourse:

$$\begin{aligned}&\text{Find } \mathbf{x} = [x_1, x_2, \ldots, x_n]^T\\&\text{to minimize}: \ \mathbf{F}(\mathbf{x}) = [E[F_1(\mathbf{x},\omega)], E[F_2(\mathbf{x},\omega)], \ldots, E[F_k(\mathbf{x},\omega)]]^T\\&\text{subject to}: \ \mathbf{x} \in D\end{aligned} \tag{5}$$

where n is the number of design variables, k is the number of objective functions, $\mathbf{x}$ is the vector of scalar design variables (of size n) and D is the feasible set. $\mathbf{F}(\mathbf{x})$ is the expected vector of objective functions (of size k). Each objective function F_i is a function of the design variables as well as the random event ω whose sample space is defined by the simulation scheme adopted for the problem. For our purpose, given any set of design variables, the expectation of each objective function is estimated by repeating the reliability analysis a fixed number of times, r. Since the design variables (in our examples as described later) are chosen before the simulation starts and are not altered/selected midway, there is no scope of recourse here; although adaptive sampling schemes that alter the design variables during simulation can be cast as an MOSO problem with recourse. The optimum solution vector of design variables is $\mathbf{x}^*$ and can be depicted both in the feasible design space D as well as the feasible criterion space $\mathbf{Z} = \{\mathbf{F}(\mathbf{x})|\mathbf{x} \in D\}$. The feasible design space could be given simply by enumeration, membership rules or by m inequality constraints and e equality constraints: $g_j(\mathbf{x}) \leqslant 0; j = 1, 2, \ldots, m$ and $h_l(\mathbf{x}) = 0; l = 1, 2, \ldots, e$. Linderoth et al. [29] has discussed about the convergence of results of sample average approximation of (5). They have also discussed about optimality conditions including first order Karush–Kuhn–Tucker (KKT) conditions, which are the necessary conditions for optimality (convexity of objective functions and constraints makes it the sufficient condition as well).

Since a single optimum point does not occur in general when dealing with MOO problems, a Pareto optimal set of solutions results [30]. A Pareto optimal point is such that further improvement in any objective will necessarily result in the deterioration of at least one of the remaining objective functions and hence result in moving away from the Pareto optimal set:

$$\begin{aligned}&\mathbf{x}^* \in \mathbf{X} \text{ is Pareto optimal iff there does not exist another point}\\&\mathbf{x} \in \mathbf{X}, \text{such that } \mathbf{F}(\mathbf{x}) \leqslant \mathbf{F}(\mathbf{x}^*), \text{and} F_i(\mathbf{x}) < F_i(\mathbf{x}^*) \text{ for at leastone } i\end{aligned} \tag{6}$$

The equivalent representation in the objective space is that of non-domination:

$$\begin{aligned}&\mathbf{F}^* \in \mathbf{Z} \text{ is non-dominated iff there does not exist another } \mathbf{F} \in \mathbf{Z}\\&\text{such that } \mathbf{F} \leqslant \mathbf{F}^* \text{ with at least one} F_i(\mathbf{x}) < F_i(\mathbf{x}^*)\end{aligned} \tag{7}$$

Of course, a user defined preference function can be used to rank order members of the Pareto set as an aid to decision making, but is beyond the scope of this work. In this paper, Pareto Optimal sets are found for SS:MCMC and IS (and its variants) solutions to a set of benchmark structural reliability problems.

Detailed discussions on various methods for solving MOO problems can be found in [30,31]. If the objective functions and constraints are convex and the feasible set is a convex set, many methods are available that can be used [32]. However when convexity cannot be demonstrated, the most robust technique for solving MOO problems is through evolutionary algorithms, among which Genetic Algorithms (GA) are the most popular choice. In addition, GA can also handle problems involving categorical and discrete variables. Evolutionary algorithms that are used for the optimization purpose need to be modified accordingly for the purpose of zeroing in on the Pareto optimal set. Due to the nature of GA it is possible that Pareto optimal points are lost during the simulation. Various strategies have been proposed to remove this problem. GA also does not involve checking whether the solutions obtained satisfy KKT conditions or not. A recent study by Tulshyan et al. [32] has incorporated a KKT based termination criterion in GA.

In this paper the NSGA II algorithm (Deb et al. [33]) is used for all the MOO problems. NSGA stands for Non-dominated Sorting Genetic Algorithm. NSGA II reduced the computational complexity associated with earlier non-dominated sorting algorithms. It uses an elitist strategy for ensuring non-dominated results. The source code available on the website of KanGAL (Kanpur Genetic Algorithms Laboratory) has been used in this work.

An Elitist strategy means that for all iterations of the GA subsequent to the first iteration the current populations are compared with the best non-dominated solution set found from the all previous iterations. It ensures faster convergence without losing the diversity of the non-dominated solution set. The concept of non-domination rank, or simply rank, is introduced to identify

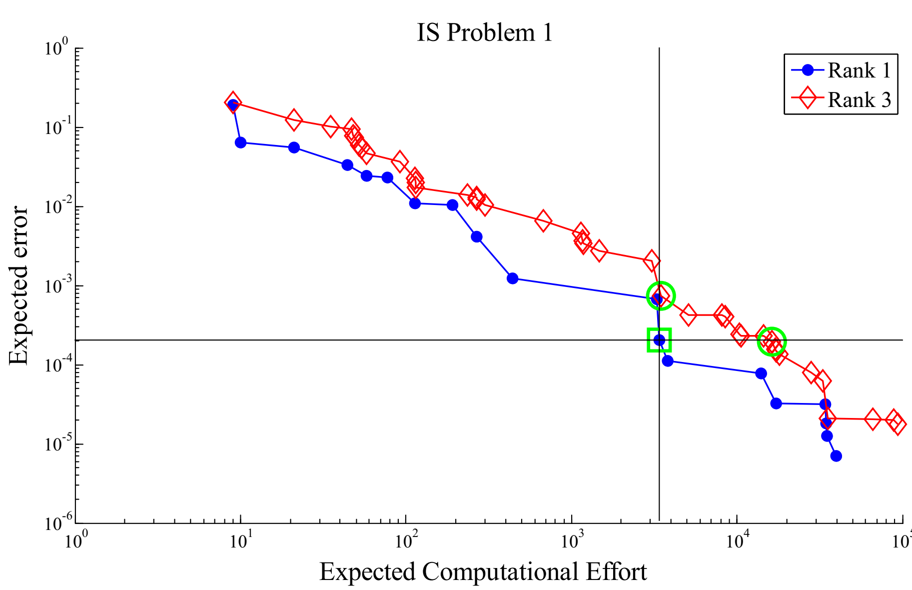

Fig. 1. Rank 1 and Rank 3 solutions from IS for Problem 1.

Table 1
Details of points marked in Fig. 1.

Rank	Number of samples	Sampling density type	Standard deviation of sampling density	Mean of sampling density	Average error	Computational effort
1	3401	Triangular	1.12	5.98	0.000204	3401
3	3460	Triangular	1.12	5.92	0.000743	3460
3	16215	Uniform	1.13	4.14	0.000194	16215

Table 2
Problem definitions.

Problem	Limit state	Variable definitions	Description	Failure probability	Failure probability obtained by	Design point from FORM
1	$g = 5 - x_1$	$x_1 \sim N(0,1)$	Linear LS	2.86e-7	Numerical integration of normal CDF	5
2	$g = 0.1(x_1 - x_2)^2 - \frac{(x_1+x_2)}{\sqrt{2}} + 2.5$	$x_1, x_2 \sim N(0,1)$	Quadratic LS with mixed term, convex LS	0.0042	MCS with 10^7 samples	(1.7678, 1.7678)
3	$g = -0.5(x_1 - x_2)^2 - \frac{(x_1+x_2)}{\sqrt{2}} + 3$	$x_1, x_2 \sim N(0,1)$	Concave LS	0.1046	MCS with 10^5 samples	(−0.815, 1.445) and (1.445, −0.815)
4	$g = 3 - x_2 + (4x_1)^4$	$x_1, x_2 \sim N(0,1)$	Highly nonlinear LS	1.8e-4	MCS with 10^8 samples	(0, 3)
5	$g = 2 + 0.015\sum_{i=1}^{9} x_i^2 - x_{10}$	$x_{1\ldots10} \sim N(0,1)$	Quadratic LS with 10 random variables	0.0165	MCS with 10^6 samples	(0, 0, 0, 0, 0,0, 0, 0, 0, 2)
6	$g = x(t) - 1.75$ $x(t)$ is the response of the SDOF system $\ddot{x}(t) + 2\zeta\omega\dot{x}(t) + \omega^2 x(t) = W(t)$ with damping ratio $\zeta = 2\%$, natural frequency $\omega = 7.85$ rad/s and spectral intensity $S = 1$ for the zero mean Gaussian white noise $W(t)$	$X_i \sim N(0,1)$ $i \in [1, 1500]$ Used to define random excitation	Dynamic problem with high dimensions	1.3e-3	MCS with 10^7 samples	NA

Table 3
Design variables.

Algorithm	Design variable	Nature of design variable	Range
SS:MCMC	Conditional probability	Discrete values	0.05, 0.1, 0.2
	Number of intermediate samples	Discrete values	100, 200, 300,. . ., 5000
	Type of proposal pdf	Categorical	Normal, Uniform, Triangular
	Standard deviation of proposal pdf	Continuous	0.5–1.5
IS	Number of Samples	Integer values	10–100,000
	Type of ISD	Categorical	Normal, Uniform, Lognormal, Gamma, Triangular
	Standard deviation of ISD	Continuous	0.5–1.5
	Mean of ISD	Continuous	Range varies from problem to problem
AIS	Number of initial failed samples	Integer values	1–100
	Number of samples generated from Kernel density	Integer values	10–10,000
	Sensitivity parameter	Continuous	0.45–0.55
AIS:MCMC	Number of initial samples	Integer values	1–100
	Number of samples generated from Kernel density	Integer values	10–1000
	Sensitivity parameter	Continuous	0.45–0.55
	Type of distribution of proposal pdf	Categorical	Normal, Uniform, Triangular
	Standard deviation of proposal pdf	Continuous	0.5–1.5
	Initial sample in failure zone	Continuous	Range varies from problem to problem. Ranges ensure that sample generated always lies in failure zone

non-dominated solutions in a given population. Lower the rank better the solution with 1 being the best rank possible, i.e., members of the Pareto optimal set. The concept of crowding distance is introduced in NSGA II to improve diversity in solution. This has reduced the computational complexity of the algorithm with respect to its predecessors [33].

The two objective functions of expected error and expected computational effort are computed from five samples (i.e., five estimates of the same reliability) in each call from NSGA II. The size is restricted to five mainly due to resource constraints; however, the scatter in the estimated means has been checked to be within acceptable limits in representative cases.

4. Numerical studies

4.1. A one-dimensional problem

Let us start with a one variable reliability problem whose solution is known. This problem is designated as Problem 1 in the suit of problems studied later. The limit state is linear in the standard normal variable:

$$g = 5 - X,\ X \sim N(0,1) \tag{8}$$

The true failure probability is 2.86×10^{-7}, a rather small number. If we were to use basic Monte Carlo simulations to estimate this

Table 4
Summary of conclusions drawn from MOSO framework.

Problem no.	Type of limit state	Best method based on location of Pareto Set in objective function space	Claims in available literature		
			FORM design point should be used as mean of ISD	Choice of proposal pdf does not affect efficiency of SS:MCMC	Sensitivity parameter of AIS should be 0.5 for optimal performance
1	Linear LS in 1 RV	IS	No	No. It does affect	Yes
2	Quadratic LS in 2 RVs with mixed term, convex LS	IS and AIS:MCMC are comparable	No	No. It does affect	No
3	Concave LS in 2 RVs	SS:MCMC for higher error, AIS for lower error	No	Not relevant	No
4	Highly nonlinear LS in 2 RVs	All of equal merit	No	No. It does affect	No
5	Quadratic LS with 10 RVs	SS:MCMC	No	No. It does affect	No
6	Dynamic problem (1500 RVs)	SS:MCMC	–	No. It does affect	–

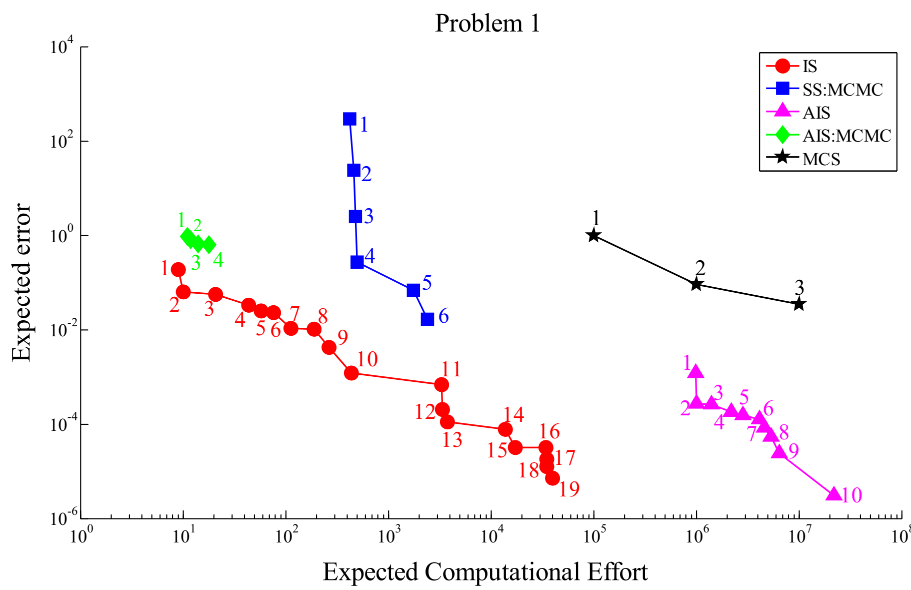

Fig. 2. Pareto sets for Problem 1.

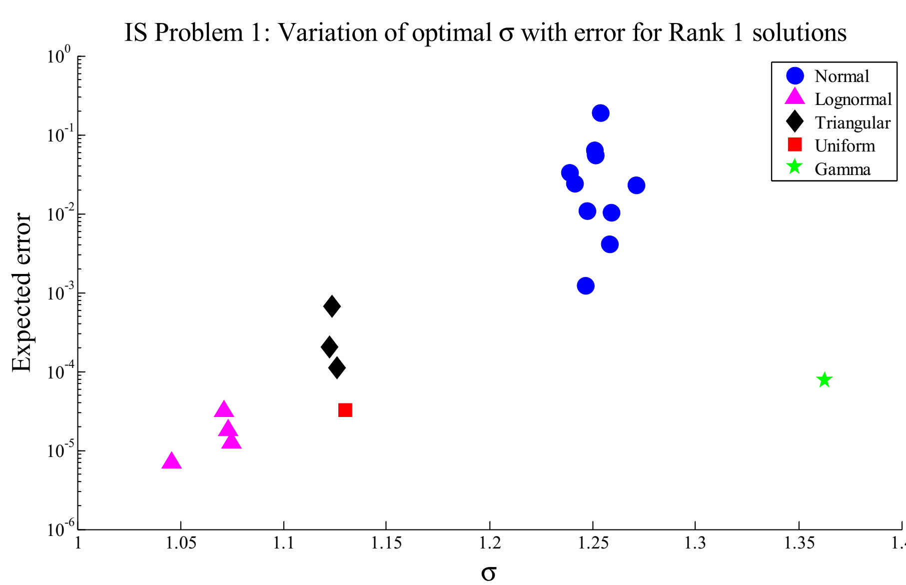

Fig. 3. Variation of optimal standard deviation of sampling density with error for IS.

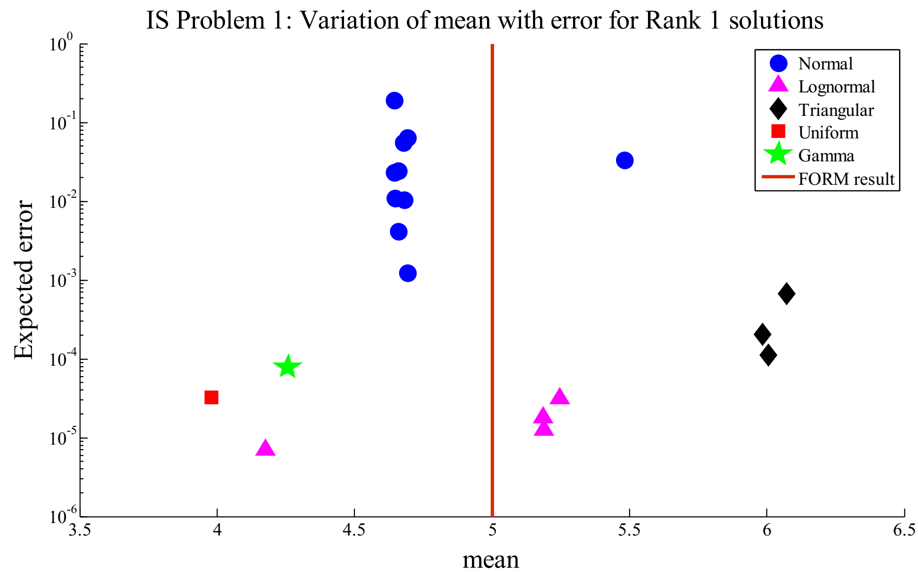

Fig. 4. Variation of mean of sampling density with error for IS.

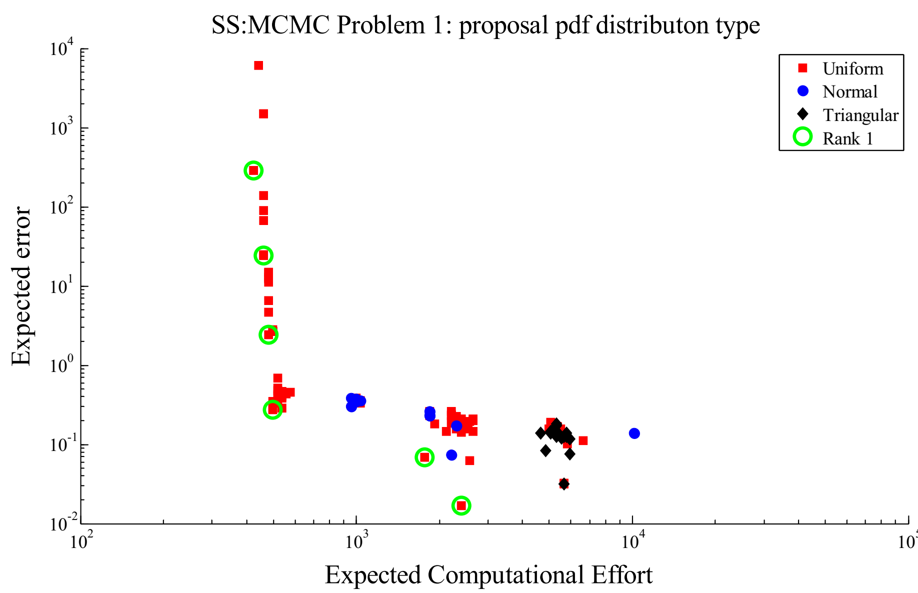

Fig. 5. Proposal pdf types of the solution set of SS:MCMC for problem 1.

number, the trade-off between effort and error would be the well-known inverse relation: c.o.v.$(\widehat{P}_f) = 1/\sqrt{P_f N}$ where P_f is the true failure probability, $\widehat{P}_f$ is the estimate, and N is the number of simulations. We will take this cue in the following, and define "average error" in all cases as the variance of the estimated failure probability, normalized by the square of the true P_f:

$$\text{Average error} = \frac{E[\widehat{P}_f - P_f]^2}{P_f^2} \tag{9}$$

which in the case of basic MCS boils down to the square of the c.o.v. of the estimate.

The other objective – average computational effort – is the number of times the vector of basic variables is generated, which for basic MCS and IS are nonrandom and user defined, but for adaptive techniques like AIS, AIS:MCMC and SS:MCMC are random in nature. As stated above, the average error and average computational effort are computed by repeating the reliability analysis $r = 5$ times for each call from NSGA II.

Let us look at the trade-off between effort and accuracy when Importance Sampling is employed to solve the reliability problem in Eq. (8). The design variables are: type, mean and variance of the sampling density and the number of trials (which is also the second objective).

Fig. 1 shows the rank 1 (i.e., Pareto) and rank 3 (i.e., inferior) solutions obtained from NSGA II for IS when applied to problem 1. Clearly rank 1 solutions are better in every respect than rank 3 solutions as can be seen by their respective positions in the objective function space. We now draw one horizontal and one vertical line through the Pareto point (3402, 0.0002) designated by the green square in the figure. This point was produced by a triangular importance sampling density centred on 5.98 and having a standard deviation of 1.12. If the mean is reduced to 5.92 keeping the other two design variables unchanged, one obtains a rank 3

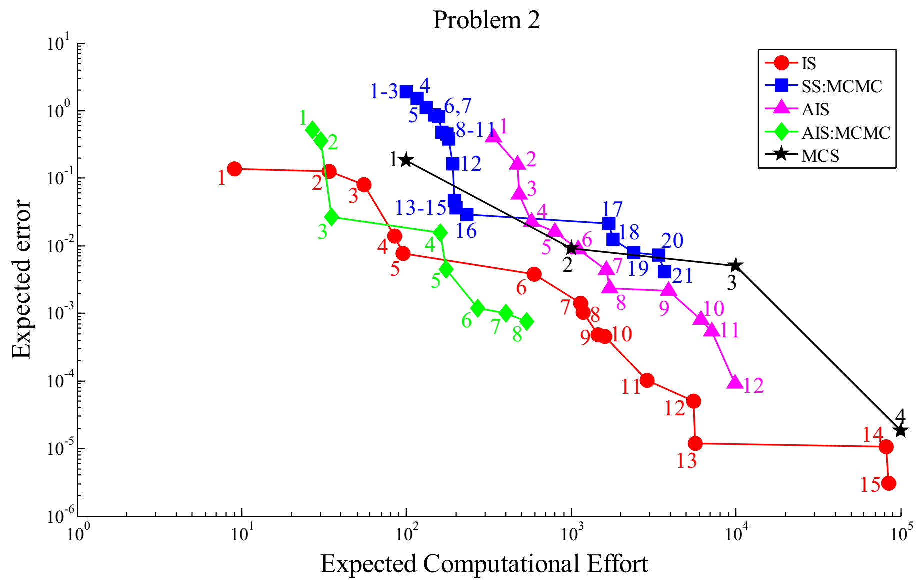

Fig. 6. Pareto sets for Problem 2.

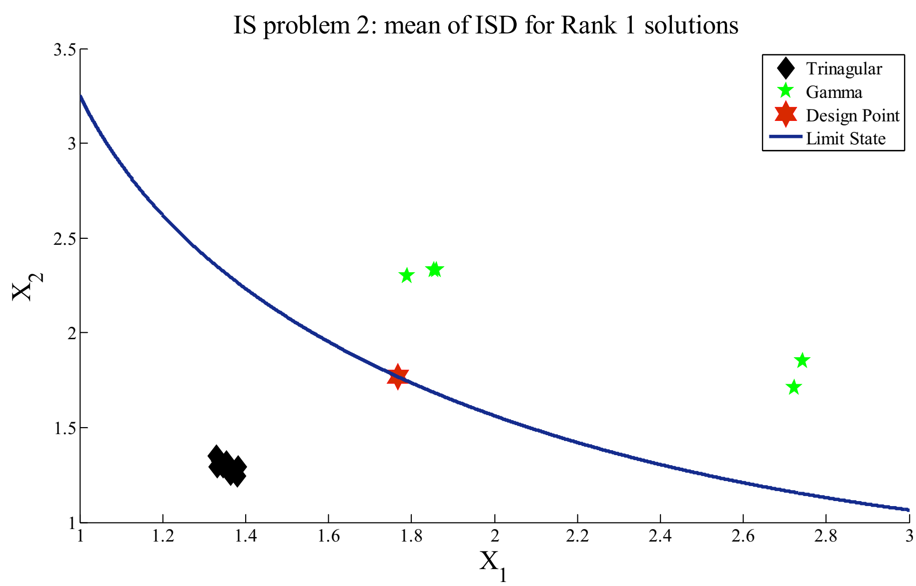

Fig. 7. Variation of mean of importance sampling density with distribution type for Problem 2.

solution that has a much larger average error. If instead, the sampling density is changed to Uniform with a smaller mean, one again obtains a rank 3 solution that has a much smaller error but now requires a much larger effort. The two rank 3 solutions are marked by green circles in the figure. The values of the design and objective variables at the three points are listed in Table 1.

Fig. 1 also gives the sort of insight that is typically not found in variance reduction studies and one that is central to the theme of this paper. Suppose we wanted to solve the reliability problem in Eq. (8) with an effort limited to about 3400 samples. Subject to the admissible range of design variables (i.e., the feasible set D), the lowest average error attainable would be about 0.0002 and no less. Further, the triangular distribution would outperform the others considered here (uniform, lognormal, normal and gamma) as the sampling density, and the mean of sampling density must not be centered at 5 which is the point of maximum likelihood, but should be closer to 6. These insights will become more relevant when we have more than one algorithm to choose from (e.g., adaptive IS, SS:MCMC etc. in addition to IS) as discussed in the next section. The position of the Pareto sets in the objective function space will enable us to comment on the efficacy of a specific method for solving different types of reliability problems. In addition to this, choice of optimal design variables can also be made on objective criteria.

4.2. A suit of benchmark problems

Table 2 lists the six different reliability problems of increasing complexity that have been selected for the present study. Problem 1 has already been defined. Problems 2–5 are adopted from Grooteman [34]. Problem 6 involving a linear SDOF has been adapted from Au and Beck [22]. The failure probabilities have not been directly taken from the sources, rather, a very large number of basic Monte Carlo Simulations (numbers given in Table 2) were

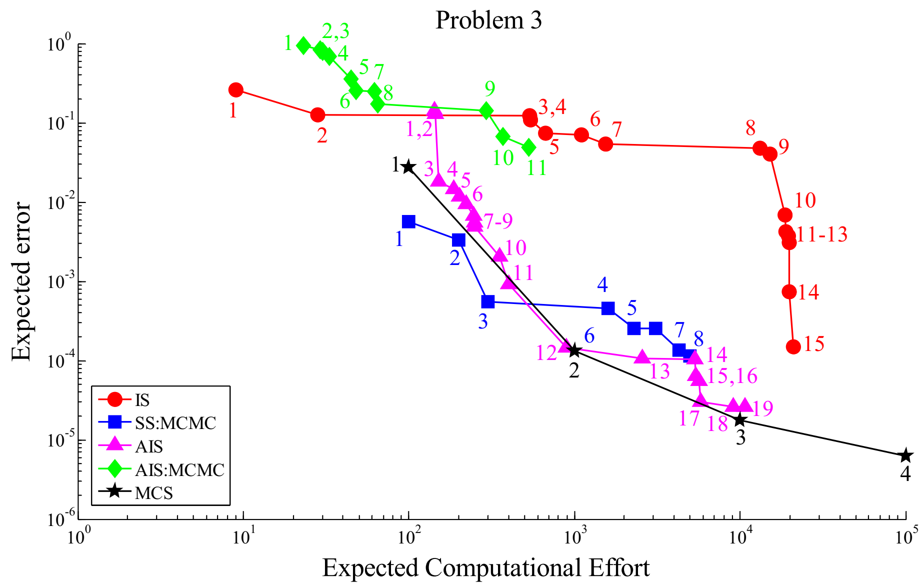

Fig. 8. Pareto sets for Problem 3.

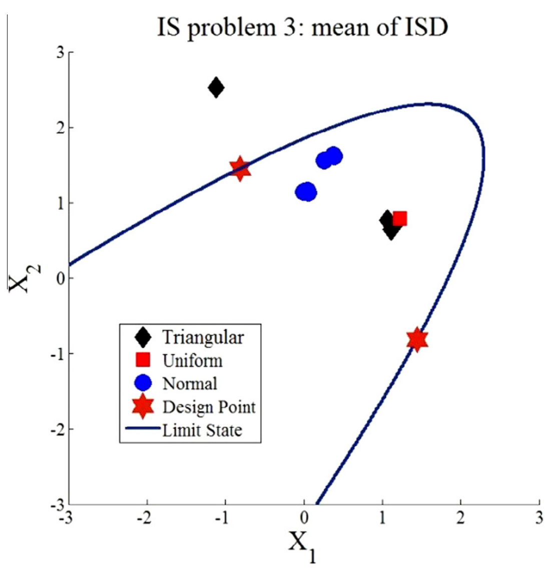

Fig. 9. Variation of mean of importance sampling density with distribution type for Problem 3.

performed to estimate the "exact" failure probabilities. These failure probabilities were then used for the average error estimation using Eq. (9).

In this study we look at Subset Simulation incorporating MCMC moves and Importance Sampling and its two variants: Adaptive Importance Sampling with MCS (i.e., AIS) and Adaptive Importance Sampling with MCMC moves (i.e., AIS:MCMC). As discussed earlier the other variants of Subset Simulation have not been looked at because the SS:MCMC is the most general purpose of them all.

Table 3 lists the design variables for each algorithm. These design variables are the parameters that govern the performance of the given algorithm and their feasible domain has been based on existing literature. Parameters that have been claimed by some researchers to not affect the performance of an algorithm have also been modeled as design variables in order to check the accuracy of such claims. For IS the mean of the sampling density is a design variable here and is not pre-evaluated by FORM.

4.3. Results and discussions

The six problems are now discussed in detail, including the behavior of the Pareto sets produced by the competing algorithms, and the role of the optimal decision variables. In the figures that follow, the Pareto sets for SS:MCMC, IS, AIS, and AIS:MCMC have been shown. In addition to this, MCS estimates have also been shown for each of the problems (for a better interpretation of the Pareto fronts). These MCS points are not obtained from any optimization process. These are the errors associated with MCS for each problem for various sample sizes (values are averages of five runs). The points in the Pareto plots have been marked with numbers. These numbers correspond to the design variables associated with each point from the Tables in the Appendix.

4.3.1. Problem 1: The 1-D Gaussian limit state

We again look at Problem 1, this time in greater detail. Recall that it is a one dimensional problem with a very low failure probability. The Pareto fronts produced by the four algorithms along with the error–effort curve of MCS are shown in Fig. 2. It is clear that IS outperforms all the other methods taken into consideration. Even though AIS can achieve a lower error (point 10) the corresponding computational effort is much higher than that of IS. It is also clear from Fig. 2 as expected that all variance reduction schemes outperform MCS.

From Table A1 in the Appendix one can observe that for IS the optimal sampling density follows a pattern. For computational effort below 1000 a normal distribution should be preferred. For mid-range computational efforts (greater than 1000 but less than 30,000) a triangular distribution is preferred. For very high computational effort region a lognormal distribution will lead to Pareto optimal performance. Thus if resources are not an issue the lognormal should be the preferred importance sampling density for this problem. Inferences about the standard deviation of the sampling density can also be drawn based on Fig. 3. It is clear that for Normal distribution a higher standard deviation is required compared to lognormal and triangular distributions. Fig. 4 shows the variation of the mean of the sampling density with error and distribution type. The FORM design point for this problem is $x_1 = 5$. It is clear that the sampling density centered away from the design point of FORM is desirable for this problem. For normal distribution a

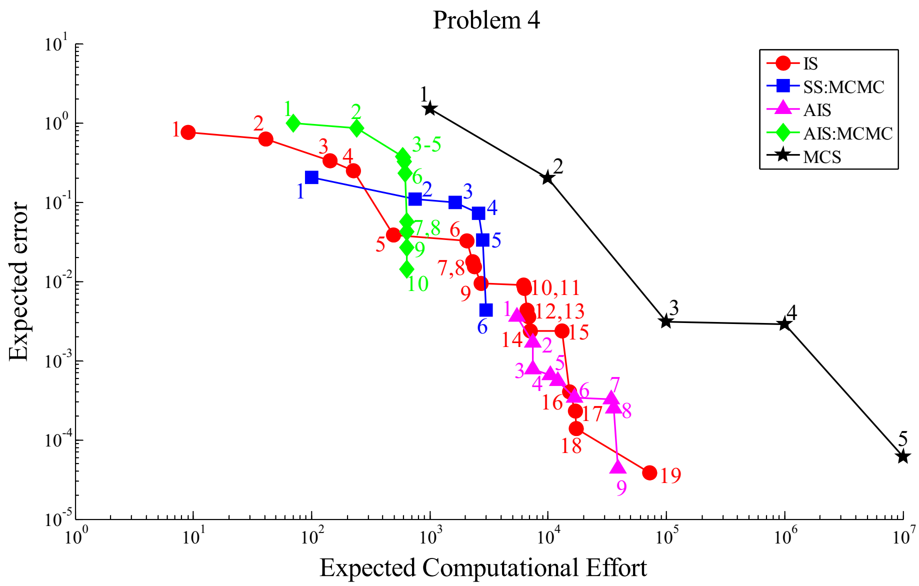

Fig. 10. Pareto sets for Problem 4.

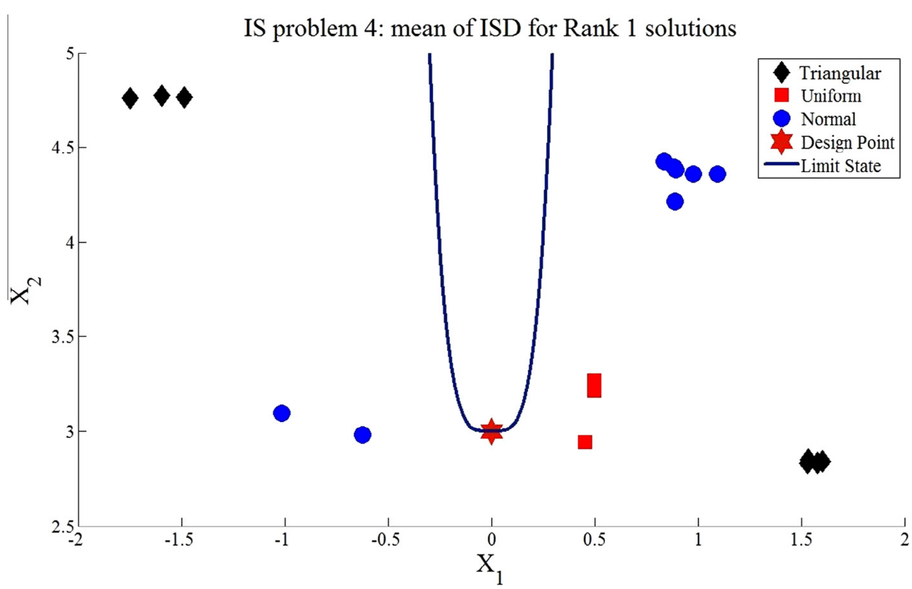

Fig. 11. Variation of mean of importance sampling density with distribution type for Problem 4.

value lower than 5 is preferred whereas for lognormal and triangular distributions it is higher than 5. In fact for uniform distribution the mean of the sampling density is near 6. These results complement those shown in Fig. 3. Since the mean of sampling density for a normal distribution is in the "safe region" for the problem, the standard deviation is higher so that more points can be generated in the failure region (that is the whole idea of IS in the first place). This combination gives a higher error but computational effort is reduced. In case of lognormal and uniform since the mean is already in the "failed region" of the problem, a lower standard deviation of the sampling density suffices.

For SS:MCMC it is clear, from Table A2 of the Appendix, that an assumed conditional probability of 0.05 and a uniform proposal pdf is required for Pareto optimal performance. This is not in accordance with the assertion, made by Au and Beck [22], that the choice of proposal pdf does not affect the performance of the algorithm. To reinforce the idea, Fig. 5 shows the types of proposal pdf for each element of the final solution set. This solution set contains both the rank 1 solutions as well as all other inferior solutions. Clearly choice of proposal pdf affects the performance as a normal or triangular proposal pdf gives inferior results.

The Pareto set of AIS located in a region of very high expected computational effort in the objective function space (Fig. 2). This is due to the very low failure probability to be estimated. Hence a very high effort is necessary to generate the required number of failed samples for construction of the kernel density. From Table A3 in the Appendix it is clear that the sensitivity parameter associated with AIS should be near 0.5 for optimal performance. This is concurrent with the claims of Ang et al. [19].

AIS:MCMC on the other hand is not a very good choice when solving one dimensional reliability problem with low failure probability compared to IS. Even though the expected computational effort is very low, lower errors than that of AIS:MCMC can be achieved by IS in the same computational regime. For this problem

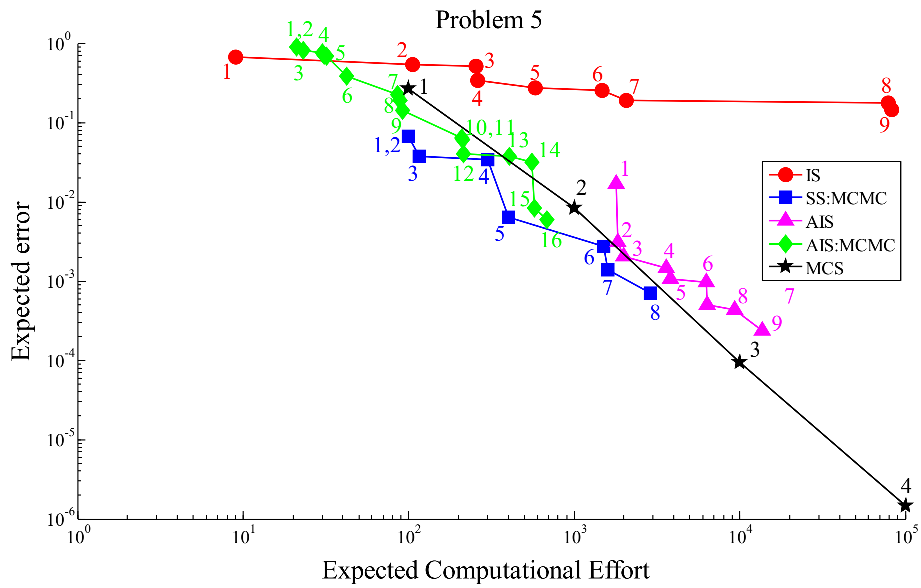

Fig. 12. Pareto sets for Problem 5.

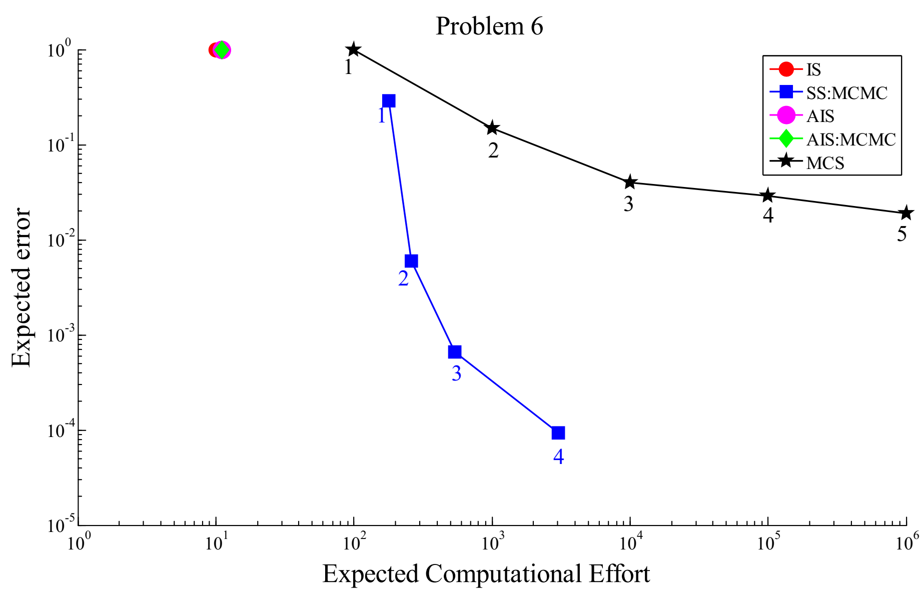

Fig. 13. Pareto sets for Problem 6.

the proposal pdf for generation of Markov Chains should be normal with a standard deviation approximately equal to 1.1 (from Table A4 of Appendix). Another conclusion that can be drawn is about the initial point of the Markov chain that needs to be generated for construction of the kernel sampling density. If lower number of samples are used for the construction of the kernel density, then the initial point of the Markov Chain is further inside the failure region of the problem (approximately $x = 6.85$). Whereas, for higher number of samples the initial point is approximately at $x = 6.3$.

4.3.2. Problem 2: Quadratic convex LS in two basic variables with mixed term

We now move up to a two dimensional reliability problem with a convex limit state. From Fig. 6 it is clear that in lower computational regime IS and AIS:MCMC are comparable and IS outperforms all the other methods in higher computational regimes. One can also observe an intersection of the Pareto fronts of SS:MCMC and AIS, from which it can be inferred that the former is better for lower computational efforts but is outperformed by the latter for high computational efforts.

For IS, for low computational effort the importance sampling density should be chosen to be triangular with a standard deviation of approximately 0.9. However, for high computational effort a gamma distribution should be used as the importance sampling density with a slightly lower standard deviation.

From Fig. 7 it becomes clear as to why for the triangular type of importance sampling density a higher standard deviation, compared to gamma distribution, is necessary. The mean of the sampling density for triangular distribution is in the safe region and hence to generate more samples in the failed region a higher standard deviation is required. The other inference that can be drawn from Fig. 8 is that the mean of the importance sapling density function is not at the design point as estimated by FORM.

Table A1
Values of design variables and objective functions of IS.

Problem number	Point number	Number of samples	Type of ISD	Mean of ISD	Standard deviation of ISD	Error	Effort
1	1	9	Normal	4.6456786	1.2537272	0.18970524	9
	2	10	Normal	4.6927389	1.2511459	0.064083938	10
	3	21	Normal	4.6775616	1.2514264	0.055460927	21
	4	44	Normal	5.4832239	1.2389811	0.033057768	44
	5	58	Normal	4.6599081	1.2413545	0.024365254	58
	6	77	Normal	4.6441141	1.2710286	0.023355392	77
	7	113	Normal	4.6486585	1.2474446	0.010773327	113
	8	191	Normal	4.6822359	1.2591364	0.010347374	191
	9	267	Normal	4.6601529	1.2583869	0.004158454	267
	10	438	Normal	4.6916038	1.2466791	0.001219458	438
	11	3282	Triangular	6.0713915	1.1235418	0.000673895	3282
	12	3401	Triangular	5.9834577	1.1224101	0.000203926	3401
	13	3803	Triangular	6.006806	1.1261828	0.000111012	3803
	14	13931	Gamma	4.2574656	1.3625637	7.73419E−05	13931
	15	17280	Uniform	3.9781445	1.1301125	3.20637E−05	17280
	16	34178	Lognormal	5.2459457	1.0712961	3.16418E−05	34178
	17	34713	Lognormal	5.1850385	1.073328	1.79278E−05	34713
	18	35096	Lognormal	5.1903744	1.0748782	1.25258E−05	35096
	19	39834	Lognormal	4.1760517	1.0455826	7.01319E−06	39834
2	1	9	Triangular	(1.35, 1.3)	0.908681	0.138485	9
	2	34	Triangular	(1.34, 1.30)	0.90482	0.125986	34
	3	55	Triangular	(1.35, 1.32)	0.907854	0.080181	55
	4	85	Triangular	(1.38, 1.29)	0.910672	0.01412	85
	5	95	Triangular	(1.33, 1.32)	0.910268	0.007631	95
	6	598	Triangular	(1.36, 1.25)	0.894342	0.0038	598
	7	1142	Triangular	(1.33, 1.29)	0.902072	0.001425	1142
	8	1181	Triangular	(1.33, 1.35)	0.903119	0.001034	1181
	9	1450	Triangular	(1.34, 1.29)	0.907434	0.000478	1450
	10	1593	Triangular	(1.38, 1.24)	0.907298	0.000462	1593
	11	2884	Gamma	(1.79, 2.30)	0.8315	0.000102	2884
	12	5535	Gamma	(1.86, 2.33)	0.874592	5.02E−05	5535
	13	5644	Gamma	(1.85, 2.33)	0.884117	1.17E−05	5644
	14	81216	Gamma	(2.72, 1.71)	0.857228	1.08E−05	81216
	15	84559	Gamma	(2.74, 1.85)	0.843898	3.05E−06	84559
3	1	9	Uniform	(1.22, 0.79)	0.979972	0.263109	9
	2	28	Triangular	(1.17, 0.72)	0.922326	0.126804	28
	3	534	Triangular	(1.17, 0.72)	0.916747	0.122151	534
	4	541	Triangular	(1.17, 0.73)	0.916748	0.108624	541
	5	671	Triangular	(1.12, 0.65)	0.910628	0.07439	671
	6	1096	Triangular	(1.11, 0.64)	1.024677	0.0704	1096
	7	1536	triangular	(1.06, 0.76)	0.987619	0.053449	1536
	8	13215	Triangular	(−1.11, 2.52)	1.464675	0.048125	13215
	9	15155	Normal	(0.26, 1.55)	0.877722	0.040284	15155
	10	18621	Normal	(0.37, 1.63)	1.02147	0.006896	18621
	11	18928	Normal	(0.39, 1.61)	1.017129	0.004264	18928
	12	19538	Normal	(−0.00, 1.13)	0.991619	0.003753	19538
	13	19718	Normal	(0.05, 1.16)	0.991728	0.003086	19718
	14	19799	Normal	(0.01, 1.15)	0.992367	0.00073	19799
	15	21078	Normal	(0.06, 1.13)	0.993092	0.00015	21078
4	1	9	Triangular	(−1.48, 4.76)	0.766507	0.76329	9
	2	41	Triangular	(−1.75, 4.76)	0.754183	0.627085	41
	3	143	Triangular	(−1.59, 4.77)	0.76777	0.333053	143
	4	225	Normal	(1.09, 4.36)	1.268906	0.249878	225
	5	491	Normal	(0.97, 4.35)	1.259412	0.037963	491
	6	2069	Normal	(0.89, 4.38)	1.32898	0.032188	2069
	7	2299	Normal	(0.88, 4.21)	1.311438	0.017871	2299
	8	2388	Normal	(0.84, 4.42)	1.337383	0.015178	2388
	9	2725	Normal	(0.88, 4.39)	1.331923	0.009515	2725
	10	6208	Triangular	(1.60, 2.84)	1.270958	0.008938	6208
	11	6311	Triangular	(1.53, 2.83)	1.282006	0.008073	6311
	12	6570	Triangular	(1.53, 2.85)	1.272277	0.004361	6570
	13	6850	Triangular	(1.53, 2.84)	1.269378	0.003549	6850
	14	7080	Triangular	(1.58, 2.83)	1.279914	0.002374	7080
	15	13103	Normal	(−1.01, 3.09)	0.656273	0.00237	13103
	16	15160	Uniform	(0.49, 3.27)	0.78861	0.000406	15160
	17	17102	Uniform	(0.45, 2.94)	1.024334	0.000232	17102
	18	17277	Uniform	(0.49, 3.21)	0.698592	0.000139	17277
	19	71640	Normal	(−0.62, 2.98)	0.985998	3.88E−05	71640
5	1	9	Triangular	(0.27,−0.7, 1.00, −0.36, −2.33, −0.59, −2.00, −2.65, −0.42, 0.52)	1.4661	0.67434	9
	2	105	Triangular	(−0.04, −0.61, 1.14,-0.52, −2.33, −0.61, −1.83, −2.36, −0.34, 0.48)	1.407515	0.534269	105

Table A1 (*continued*)

Problem number	Point number	Number of samples	Type of ISD	Mean of ISD	Standard deviation of ISD	Error	Effort
	3	256	Triangular	(−0.08, −0.7, 1.07,-0.39, −2.33, −0.66, −1.71, −2.39, −0.36, 0.39)	1.466095	0.512324	256
	4	262	Triangular	(−0.03, −0.71, 0.99,-0.45, −2.33, −0.69, −1.67, −2.44, −0.31, 0.48)	1.434387	0.341659	262
	5	583	Triangular	(0.03, −0.71, 1.01,-0.44, −2.33, −0.72, −1.86,−2.53, −0.41, 0.53)	1.465006	0.273517	583
	6	1476	Triangular	(−0.08, −0.68, 1.19,-0.46, −2.3, −0.68, −1.83, −2.38, −0.31, 0.41)	1.468804	0.256581	1476
	7	2052	Triangular	(−0.02, −0.61, 1.05,-0.42, −2.25, −0.72, −1.86, −2.46, −0.46, 0.67)	1.464632	0.188725	2052
	8	78826	Triangular	(2.89, −2.07, 0.32, −0.22, 2.06, 1.01, 1.76, 0.58, 1.24, 0.24)	1.39512	0.177728	78826
	9	82367	Triangular	(2.95, −2.08, 0.30, −0.22, 2.05, 0.91, 1.94, 0.61, 1.22, 0.23)	1.397727	0.144961	82367

From the values of the design variables of SS:MCMC it is clear again that the type of proposal pdf does affect the performance of the algorithm. For very high computational effort a triangular distribution should be preferred. For other computational effort regimes a normal or uniform distribution suffices, with appropriate choice of standard deviation and conditional probability. Another observation that can be made from the Table A2 in Appendix is that for higher conditional probabilities the standard deviations are lower. This is true because higher conditional probabilities make the length of intermediate Markov Chains shorter. Hence, to reduce repetition in the Markov chain, to ensure effective spanning of the variable space, the standard deviation of the proposal pdf is lower.

4.3.3. Problem 3: Quadratic concave LS in two basic variables with mixed term

In contrast to Problem 2, Problem 3 is a two dimensional concave reliability problem with a high failure probability. From Fig. 8 it is clear that at low computational effort SS:MCMC outperforms all other methods. Since there is an intersection of the Pareto fronts of AIS and SS:MCMC, the former outperforms the latter when the computational effort is high. Another conclusion that we can draw is that AIS performs best when solving concave limit states out of all the variants of Importance Sampling. IS performs significantly poorly in Problem 3, the reason being the presence of two equidistant design points on the limit state on two different sides of the origin (Fig. 9).

Since the failure probability is very high for this Problem the results of SS:MCMC are same as that of MCS i.e. reliability estimates were found in the first step of the SS:MCMC algorithm. Hence the values of the design parameters, pertaining to the proposal pdf, obtained have no significance as they are never actually used. Also the results of MCS almost overlaps the Pareto set of SS:MCMC. For AIS again the sensitivity parameter need not be 0.5 for optimal performance as can be seen from Table A3.

4.3.4. Problem 4: Highly nonlinear LS in two basic variables

Problem 4 is a highly nonlinear reliability problem in 2 dimensions with a low failure probability compared to all the previous problems except for the first. There are a lot of intersections between the Pareto fronts of the methods (Fig. 10) suggesting that for this problem all methods are of nearly equal merit. Since the failure probability is low in this case, the position of the MCS result demonstrates the already known advantages of variance reduction schemes. Another interesting observation that can be made is that even though the failure probability is low AIS is not completely outperformed by the other algorithms as was the case in Problem 1.

From Fig. 11, we again observe that the design point obtained from FORM does not guarantee efficiency for IS. For SS:MCMC, in case of low computational efforts the conditional failure probability should be taken to be 0.05. For higher computational efforts it should be taken as 0.2. The standard deviation of the proposal pdf should also be low for lower number of intermediate samples used. For higher number of intermediate samples a higher standard deviation of the proposal pdf is preferred. This can be explained on the basis that when the number of intermediate samples is low the Markov Chains that are generated are also shorter. Hence in such a short sample if one wants to effectively span the variable space, one needs to reduce repetition in a Markov Chain (which can be achieved by reducing rejection rate; standard deviation of the proposal pdf affects the rejection rate [22]).

4.3.5. Problem 5: Quadratic LS with 10 basic variables

Problem 5 is a nonlinear reliability problem in ten dimensions with a higher failure probability compared to Problems 1 and 4 and we now start seeing the benefit of SS:MCMC. From Fig. 12 it is clear that SS:MCMC outperforms all the other techniques. The problem of high dimensionality seems to affect IS more than its variants as can be seen from the relative positions of the corresponding Pareto sets (the error associated with IS is very high compared to its variants). This is because AIS and AIS:MCMC overcome the issue of dimensionality to an extent (i.e., for not very high dimensional problems) by combining an initial MCS based estimate with estimate from kernel sampling density and by using MCMC simulations for constructing a better kernel sampling density. Also due to the high dimensional nature of the problem the variance reduction techniques do not necessarily perform better than MCS as can be seen from Fig. 12.

From Table A1 of the Appendix it is clear that for this problem the type of sampling density for IS should be triangular. The standard deviation can afford to be low when the computational effort is high such the vector space is amply covered near the limit state. As in the earlier problems, the mean of the sampling density is not equal to the FORM design point (Table 2).

For SS:MCMC the first two points happen to be MCS solutions (i.e., no intermediate failure level was necessary). The proposal pdf changes from normal to triangular as we move from lower to higher computational regime. At low computational effort the conditional probability is high along with a low standard deviation of the proposal pdf. This may lead to poor results due to shorter Markov Chains with a high rejection rate. However for this problem these drawbacks do not affect the performance as the failure probability is high.

From Table A3 of the Appendix, the sensitivity parameter associated with AIS takes values slightly higher than 0.5 for lower computational effort regimes. However for higher computational effort the sensitivity parameters take values close to 0.5 as

Table A2
Values of design variables and objective functions for SS:MCMC.

Problem number	Point number	Number of intermediate samples	Conditional probability	Type of proposal pdf	Standard deviation of proposal pdf	Error	Effort
1	1	100	0.05	Uniform	0.872192	288.3444	423
	2	100	0.05	Uniform	0.868527	24.1341	461
	3	100	0.05	Uniform	0.871299	2.435486	480
	4	100	0.05	Uniform	0.89924	0.271343	499
	5	400	0.05	Uniform	0.985234	0.068458	1768
	6	500	0.05	Uniform	0.778918	0.016743	2400
2	1	100	0.2	Normal	0.654107	1.90703	100
	2	100	0.2	Normal	0.654482	1.90703	100
	3	100	0.05	Uniform	1.391493	1.90703	100
	4	100	0.2	Normal	0.654107	1.526077	116
	5	100	0.2	Normal	0.638203	1.130762	132
	6	100	0.2	Normal	0.652807	0.854875	148
	7	100	0.05	Uniform	1.330389	0.816327	157
	8	100	0.2	Normal	0.63342	0.473923	164
	9	100	0.2	Uniform	0.694711	0.473923	164
	10	100	0.05	Uniform	1.393558	0.447279	176
	11	100	0.2	Normal	0.774501	0.386848	180
	12	100	0.1	Normal	0.62033	0.163265	190
	13	100	0.05	Normal	0.768382	0.047052	195
	14	200	0.05	Uniform	1.386137	0.036281	200
	15	200	0.05	Uniform	1.392416	0.036281	200
	16	200	0.2	Normal	0.664094	0.029478	232
	17	1700	0.1	Uniform	0.615838	0.021099	1700
	18	1800	0.1	Uniform	0.644399	0.012486	1800
	19	2400	0.2	Uniform	0.558979	0.007937	2400
	20	3400	0.05	Triangular	0.92761	0.007368	3400
	21	3700	0.2	Triangular	1.147179	0.004123	3700
3	1	100	0.05	Triangular	0.849076	0.00559	100
	2	200	0.05	Triangular	0.811386	0.003287	200
	3	300	0.2	Normal	1.005077	0.000553	300
	4	1600	0.1	Triangular	0.89679	0.000452	1600
	5	2300	0.05	Uniform	0.978379	0.000255	2300
	6	3100	0.2	Uniform	1.423807	0.000251	3100
	7	4300	0.1	Normal	1.281347	0.000137	4300
	8	5000	0.1	Uniform	1.281838	0.000113	5000
4	1	100	0.05	Uniform	0.823592	0.205642	100
	2	300	0.05	Triangular	0.851191	0.108786	756
	3	700	0.05	Normal	1.314344	0.098411	1631
	4	1200	0.05	Triangular	1.460101	0.071344	2568
	5	1200	0.05	Triangular	1.483042	0.032871	2796
	6	3000	0.2	Uniform	1.459192	0.004294	3000
5	1	100	0.2	Normal	0.650902	0.067034	100
	2	100	0.2	Normal	0.641669	0.067034	100
	3	100	0.2	Normal	0.626987	0.037649	116
	4	300	0.2	Normal	0.613758	0.034384	300
	5	400	0.2	Normal	0.509807	0.006428	400
	6	1500	0.2	Uniform	0.61886	0.002714	1500
	7	1600	0.1	Normal	0.6731	0.001377	1600
	8	2900	0.1	Triangular	0.961669	0.0007	2900
6	1	100	0.2	Normal	0.62	0.289941	180
	2	100	0.2	Normal	0.62	0.005917	260
	3	300	0.2	Normal	0.62	0.000657	540
	4	1600	0.1	Normal	0.52	9.25E−05	3040

suggested by Ang et al. [19]. Hence, it is not necessary that a sensitivity parameter of 0.5 give optimal performance for AIS.

For AIS:MCMC a triangular pdf should be used as the proposal pdf whose standard deviation should be below 1 for low computational efforts and vice versa. From Tables A4 of the Appendix, it can be seen that this increase of standard deviation takes place with the increase in the number of failed samples used to construct the kernel sampling density. For lower number of failed samples, one would want to reduce the number of repetitions in a Markov chain (i.e., generate more unique samples) to ensure that a greater area of the vector space is spanned to improve the kernel sampling density.

4.3.6. Problem 6: Time dependent reliability problem

We finally arrive at a first passage problem of structural dynamics where the basic variable space, if discretized, runs into thousands (1500 in this case). The random variables are the sequence of i.i.d. standard normal random variables that generate the white noise inputs as $\sqrt{2\pi S/\Delta t}\mathbf{X}_k$ (where $\mathbf{X}_k$ are the elements of the random vector and Δt are time steps for the analysis). The linear oscillator is observed for 30 s with time steps of 0.02 s. Failure is defined as the event when the displacement at any time instant crosses a certain threshold for the first time (in this case 1.75). IS and its variants clearly fail for this problem as can be seen from the corresponding degenerate points in Fig. 13. The

Table A3
Values of design variables and objective functions for AIS.

Problem number	Point number	Number of samples for kernel sampling density generation	Number of samples generated from kernel sampling density	Sensitivity parameter	Error	Effort
1	1	1	1754	0.495159	0.001189	994095.6
	2	2	3175	0.455696	0.000271	1001908
	3	1	2527	0.498147	0.000258	1417061
	4	1	2486	0.495926	0.000179	2187198
	5	1	2549	0.4984	0.000152	2836061
	6	1	8767	0.542907	0.000126	4122489
	7	2	2309	0.49574	8.25E−05	4594386
	8	2	2438	0.498233	5.31E−05	5406590
	9	4	8458	0.521331	2.34E−05	6406349
	10	6	7757	0.484378	3.01E−06	21868249
2	1	8	41	0.477386	0.406244	339
	2	6	127	0.480078	0.160165	473.8
	3	10	48	0.477212	0.056981	482
	4	10	80	0.481278	0.022536	577.2
	5	10	100	0.481418	0.016065	796.4
	6	13	353	0.524966	0.008915	1096
	7	27	262	0.5427	0.004318	1637
	8	15	844	0.528552	0.002363	1720.6
	9	42	1484	0.45	0.002188	3915.8
	10	89	923	0.477736	0.000792	6107.8
	11	69	3487	0.506608	0.000536	7152.4
	12	78	5291	0.500398	8.98E−05	9877
3	1	17	13	0.519378	0.140463	142.6
	2	17	10	0.468221	0.128509	143.8
	3	17	10	0.520771	0.018054	152
	4	18	13	0.469296	0.014444	186
	5	20	12	0.519432	0.011595	203
	6	15	87	0.47435	0.009523	220.8
	7	19	91	0.519443	0.006733	247.6
	8	18	79	0.52077	0.005487	249
	9	17	101	0.469268	0.004893	249.2
	10	17	201	0.469833	0.002062	353
	11	19	213	0.521014	0.000915	402.2
	12	49	489	0.481898	0.000147	889.8
	13	65	1947	0.46832	0.000105	2572.2
	14	69	4679	0.486744	0.000102	5356.2
	15	70	4771	0.48723	6.37E−05	5392.8
	16	70	4985	0.487799	5.56E−05	5662.2
	17	70	5143	0.487213	3.03E−05	5775.2
	18	75	8467	0.475199	2.6E−05	9150.4
	19	82	10000	0.473872	2.58E−05	10712.8
4	1	2	3244	0.471599	0.003579	5409.6
	2	2	3575	0.475496	0.001702	7383.8
	3	1	3469	0.475488	0.000771	7397.4
	4	2	5345	0.457016	0.000658	10511.2
	5	3	5276	0.452708	0.000547	11994
	6	2	7748	0.504669	0.00034	16645.4
	7	7	6714	0.523636	0.000324	34428.2
	8	7	6662	0.525295	0.000248	35813
	9	7	6607	0.525145	4.34E−05	38764.8
5	1	20	698	0.537781	0.01699	1794.6
	2	20	853	0.540744	0.003097	1844.6
	3	20	845	0.537249	0.002032	1988.6
	4	35	1646	0.472218	0.001455	3578
	5	35	1646	0.498262	0.00106	3795.4
	6	65	2671	0.502838	0.00096	6302.6
	7	67	2448	0.503155	0.000504	6374.2
	8	36	7263	0.533459	0.000436	9369.4
	9	90	7912	0.497429	0.000238	13645

deficiencies in IS and its variants when solving problems of high dimensions are discussed by Au and Beck [21]. For SS:MCMC it is evident that the proposal pdf in this case should be a normal distribution. Also the conditional probability should be set at 0.2 for lower computational efforts. In case of a high computational effort requirement the conditional probability should be taken as 0.1.

5. Summary and conclusions

A new way of looking at the error versus efficiency issue for simulation techniques in structural reliability has been presented. A Multi-objective Stochastic Optimization (MOSO) problem is formulated using expected values of error and computational effort as objective functions. The design variables used are parameters of

Table A4
Values of design variables and objective functions for AIS:MCMC.

Problem number	Point number	Number of samples for kernel sampling density generation	Number of samples generated from kernel sampling density	Sensitivity parameter	Type of proposal pdf	Standard deviation of the proposal pdf	Assumed initial point for MCMC	Error	Effort
1	1	1	10	0.497854	Normal	1.099332	6.86	0.923874	11
	2	2	10	0.502367	Normal	1.121369	6.87	0.801141	12
	3	4	10	0.493516	Normal	1.114545	6.85	0.664555	14
	4	8	10	0.49388	Normal	1.125594	6.30	0.621683	18
2	1	17	10	0.456592	Triangular	0.839965	(4.58, 6.3)	0.524513	27
	2	20	10	0.456186	Triangular	0.836154	(4.58, 6.32)	0.356476	30
	3	25	10	0.451716	Triangular	0.839121	(4.64, 6.27)	0.026809	35
	4	63	98	0.515117	Triangular	1.5	(5.97, 5.09)	0.015538	161
	5	64	109	0.515476	Triangular	1.5	(5.97, 5.12)	0.004526	173
	6	91	179	0.504567	Normal	0.533679	(3.33, 2.85)	0.001177	270
	7	75	325	0.500033	Normal	1.034882	(4.87, 6.29)	0.001019	400
	8	65	469	0.464899	Normal	1.402708	(8.44, 8.96)	0.000766	534
3	1	13	10	0.543481	Uniform	0.824508	(7.17, 7.13)	0.945733	23
	2	14	15	0.542285	Uniform	0.816165	(7.15, 7.13)	0.831238	29
	3	15	15	0.548042	Uniform	0.839965	(7.31, 6.98)	0.803506	30
	4	18	15	0.541784	Uniform	0.816347	(7.11, 7.11)	0.693503	33
	5	35	10	0.520045	Triangular	0.952588	(5.41, 2.6)	0.360405	45
	6	38	10	0.519675	Triangular	0.952421	(5.27, 2.63)	0.254229	48
	7	36	26	0.526319	Triangular	0.949296	(5.44, 2.92)	0.249605	62
	8	37	28	0.525093	Triangular	0.947111	(5.44, 2.88)	0.174339	65
	9	62	232	0.455829	Triangular	0.744231	(8.9, 9.44)	0.14108	294
	10	50	322	0.47015	Triangular	1.473298	(9.79, 2.6)	0.066579	372
	11	100	433	0.539711	Normal	0.77086	(4.99, 8.86)	0.04905	533
4	1	35	35	0.484946	Triangular	0.815037	(0.52, 40.13)	1	70
	2	91	148	0.467082	Uniform	1.302877	(0.96, 47.37)	0.849952	239
	3	81	509	0.53318	Uniform	0.909355	(0.65, 18.72)	0.372748	590
	4	80	515	0.537693	Uniform	0.907958	(0.62, 18.01)	0.368427	595
	5	81	526	0.533194	Uniform	0.907719	(0.62, 18.59)	0.322966	607
	6	81	532	0.533252	Uniform	0.904696	(0.61, 13.04)	0.230746	613
	7	77	557	0.532677	Uniform	0.908145	(0.59, 13.33)	0.056	634
	8	79	556	0.533167	Uniform	0.90798	(0.61, 13.43)	0.042022	635
	9	81	557	0.532988	Uniform	0.905341	(0.61, 13.52)	0.026955	638
	10	81	559	0.53284	Uniform	0.904649	(0.61, 12.92)	0.014285	640
5	1	11	10	0.538267	Triangular	0.5	(0.5, 0.58, 0.69, 0.35, 0.12, 0.7, 0.6, 0.6, 0.7, 0.69, 2.19)	0.902056	21
	2	13	10	0.531761	Triangular	0.632347	(0.63, 0.55, 0.7, 0.32, 0.16, 0.66, 0.6, 0.56, 0.69, 0.74, 2.13)	0.805914	23
	3	17	13	0.537631	Triangular	0.568914	(0.57, 0.57, 0.69, 0.33, 0.13, −.7, 0.59, 0.57, 0.72, 0.68, 2.13)	0.748286	30
	4	18	13	0.535769	Triangular	0.554731	(0.55, 0.57, 0.69, 0.33, 0.13, 0.7, 0.58, 0.59, 0.71, 0.67, 2.13)	0.710108	31
	5	20	12	0.540452	Triangular	0.596323	(0.59, 0.6, 0.7, 0.33, 0.13, 0.7, 0.6, 0.55, 0.71, 0.68, 2.15)	0.693119	32
	6	17	25	0.541125	Triangular	0.578445	(0.58, 0.56, 0.71, 0.33, 0.11, 0.71, 0.62, 0.54, 0.69, 0.71, 2.13)	0.379753	42
	7	69	17	0.478137	Triangular	1.164186	(1.16, 0.58, 0.78, 0.54, 0.2, 0.14, 0.9, 0.04, 0.09, 0.53, 2.41)	0.225309	86
	8	69	20	0.475102	Triangular	1.165157	(1.16, 0.59, 0.77, 0.48, 0.2, 0.15, 0.88, 0.04, 0.09, 0.52, 2.4)	0.188291	89
	9	75	17	0.4784	Triangular	1.156858	(1.15, 0.58, 0.77, 0.53, 0.21, 0.15, 0.86, 0.04, 0.09, 0.52, 2.43, 0.14)	0.143223	92
	10	82	128	0.526842	Triangular	1.314675	(1.31, 0.41, 0.78, 0.98, 0.45, 0.03, 0.54, 0.92, 0.99, 0.4, 3.79, 0.06)	0.064399	210
	11	82	130	0.523807	Triangular	1.315647	(1.31, 0.42, 0.78, 0.92, 0.45, 0.04, 0.51, 0.92, 0.99, 0.4, 3.79, 0.06)	0.060582	212
	12	84	130	0.522786	Triangular	1.345572	(1.34, 0.41, 0.77, 0.93, 0.45, 0.04, 0.51, 0.92, 0.97, 0.43, 3.81, 0.04)	0.040614	214

Table A4 (*continued*)

Problem number	Point number	Number of samples for kernel sampling density generation	Number of samples generated from kernel sampling density	Sensitivity parameter	Type of proposal pdf	Standard deviation of the proposal pdf	Assumed initial point for MCMC	Error	Effort
	13	50	358	0.451978	Triangular	1.009265	(1, 0.89, 0.62, 0.68, 0.61, 0.31, 0.13, 0.6, 0.37, 0.92, 2.62, 0.03)	0.037842	408
	14	93	460	0.502359	Triangular	1.442365	(1.44, 0.06, 0.53, 0.02, 0.67, 0.81, 0.45, 0.46, 0.31, 0.07, 4.88, 0.03)	0.031869	553
	15	94	480	0.502483	Triangular	1.456185	(1.45, 0.04, 0.53, 0.03, 0.69, 0.81, 0.42, 0.46, 0.31, 0.09, 4.76, 0)	0.008399	574
	16	46	639	0.519284	Triangular	1.399366	(1.39, 0.28, 0.59, 0.63, 0.11, 0.95, 0.69, 0.93, 0.22, 0.46, 2.41, 0)	0.005959	685

Table A5
Values of error and effort of MCS.

Problem number	Point number	Error	Effort
1	1	1	100000
	2	0.091379	1000000
	3	0.034637	10000000
2	1	0.183673	100
	2	0.00907	1000
	3	0.005102	10000
	4	1.84E−05	100000
3	1	0.027672	100
	2	0.000132	1000
	3	1.77E−05	10000
	4	6.18E−06	100000
4	1	1.493827	1000
	2	0.197531	10000
	3	0.003086	100000
	4	0.002844	1000000
	5	6.05E−05	10000000
5	1	0.26538108	100
	2	0.008264463	1000
	3	9.40312E−05	10000
	4	1.46924E−06	100000
6	1	1	100
	2	0.1479	1000
	3	0.04	10000
	4	0.0286	100000
	5	0.0189	1000000

the algorithm that in previous research have been shown to affect the performance of the algorithms; as such they are variously of continuous, discrete and categorial types and the multiobjective problem has been solved using Genetic Algorithms.

We have considered four variance reducing algorithms in this work: subset simulations with MCMC moves (SS:MCMC), Importance Sampling (IS), Adaptive Importance Sampling (AIS) and AIS with MCMC moves (AIS:MCMC) and have applied them to six different types of reliability problems of increasing complexity. Pareto sets produced by the four algorithms for each problem along with the optimal design variables for each Pareto point have been obtained. These Pareto sets provide insight into which algorithm should be used for different reliability problems and what the best attainable performance is from a given algorithm for a given class of reliability problem. Suggestions about the design parameters for each algorithm have also been made to attain optimal computational effort and accuracy for each class of problem.

Table 4 summarizes the general conclusions from this work. In addition, the common knowledge that IS is the best algorithm for solving simple problems as long as the limit state is not concave has been reinforced. AIS requires a very high computational effort when the failure probability estimate is very low suggesting that its application should be limited to problems with relatively high failure probability. The advantage of SS:MCMC is realized while solving high dimensional reliability problems as in structural dynamics and for such problems the Pareto set for IS and its variants become degenerate.

The proposed approach is able to capture the limits of performance for each algorithm for the class of problems considered in this work. This work will be extended to complex dynamic systems and other recent methods. The vast information that one can acquire about the performance of a variance reducing algorithm along with the corresponding design parameters from this approach, will be crucial in deciding the best approach for solving a given reliability problem when the user is constrained by the available computational resources.

Appendix .

(See Tables A1–A5.)

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