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ORIGINAL PAPER

Non-iterative eigenstructure assignment technique for finite element model updating

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Abstract A novel control theory-based eigenstructure assignment (ESA) technique is employed to update the finite element model (FEM) of a linear time-invariant system. The proposed method uses state feedback to produce the gain matrix which in turn updates the existing system matrices through simultaneous assignment of eigenvalue-vector pairs in the FEM generated system matrices. However, unlike general control technique no external input is used to control the system. A subspace identification algorithm is applied to develop the state space model and the eigenstructure of its state matrix is used as the target for the ESA algorithm to update the FEM. Unlike most FEM updating algorithms which use optimization techniques, this method is not iterative (and hence computationally less expensive) as the state matrix gets updated just once yielding the single best possible solution to match the desired eigenstructure. Furthermore, it naturally preserves the basic exploitable properties like symmetry, sparsity, positive definiteness of the updated matrices. The method is first validated numerically on a four-story shear frame subjected to ambient vibrations. Following this, a two-story one-bay aluminium frame is subjected to a suite of excitations in the laboratory, and the response time histories are put through canonical variate analysis algorithm to yield the real system state space model, and hence the modal parameters. Noise is taken care of by singular value decomposition of the signal and performing ensemble averaging of the state matrix in its companion form. A very close conformation of the updated model is observed when compared to the modal parameters extracted from structural response.

Keywords Finite element model updating · Control theory · Eigenstructure assignment · Subspace identification

1 Introduction

Finite element modeling (FEM) is essentially a process of idealizing an infinite degrees of freedom (DOF) system into a finite DOF at the cost of accuracy. These idealized models thus deviate from the real structural systems owing to assumptions made in regard to boundary conditions, model order and material and geometric parameters. Model updating is therefore undertaken, ideally with experimental cross checking, thus that it can predict the actual response with better accuracy. The extensive use of experimentally validated FEMs of complex structure subjected to dynamic loading have always proved to be a reliable way to verify structural performance for any kind of load cases. In order to make the predictor model more reliable systematic improvements through calibrating its properties with the real structure are needed. For that very reason FEM updating has always been an interesting subject of research for last few decades [2]. Unlike system identification, which mostly models the system as black box using almost no knowledge about the system, FE model updating can be classified as a gray box modelling approach. In FE model updating a physical model is derived using prior knowledge about the physics of the system and then updated using previously identified system properties. Thus the

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updated model still holds its physical significance while its response conforms to the real system response. Experimentally measured data (e.g., response history, modal properties) obtained from real system is an important component in this procedure as it helps to set the target for the FE model to achieve.

Different modal testing techniques [12] have been extensively used for modal parameter estimation of the real structures. The advancement in the field of digital signal processing (DSP) and fast Fourier transform technique (FFT) frequency domain decomposition (FDD) method [6] of modal testing became popular. However, requirement of large amount of data to properly identify the modal properties, deleterious effect of windowing on damping estimation and poor performance with the non-stationary signals makes FDD less effective in the identification of real field structures. There are other methods related to time or frequency domains to identify a system through measured data in literature. Of these wavelet transform [11] is a combination of time and frequency domain. Examples for time domain methods include Ibrahim time domain method (ITD) [15], complex exponentials method [23], polyreference method [1], eigensystem realization algorithm (ERA) [17]. Eigensystem realization theory is an approach of identification based on linear dynamic systems analysis and control [28]. This method produces a state space model of sufficiently higher order to describe system dynamics relating inputs and outputs through some internal variable termed as states, which may or may not be physically interpreted. In this method, identification of the model is done though fitting a state space model of the damped dynamics of the system [17–19, 26].

To obtain a better predictor model with physical significance researchers tried to utilize identified system properties through FEM updating [13, 16]. Existing techniques to update any FEM can be grouped in four different categories, namely- direct methods, indirect methods, control theory-based techniques and probabilistic approach. Eigenstructure assignment-based FEM updating can come under the category of control theory-based techniques. Eigenvalue assignment or pole placement has always been an interesting field of research for control engineers. Generally pole placement techniques are used with an objective to control a system with minimum control effort possible. There are several pole placement techniques exist in literature. Arbitrary assignment of eigenvalues for a closed loop system has been discussed by Wonham [33]. Moore [25] was the first person to identify the flexibility offered by state feedback in multivariate systems beyond closed loop eigenvalue assignment. He further demonstrated in his paper that a specific number of elements of each eigenvector of a closed loop MIMO system can be freely assigned. Kautsky et al. [21], Srinathkumar [30] discussed robust pole assignment technique in linear time invariant system. Several other researchers (Garrard et al. [14], Sobel et al. [29]) also developed algorithm to place eigenstructure for closed loop system.

ESA for model updating is, however, relatively new field. Generally vibration data and modal properties have been used for model updating in most of the literature [4]. Quadratic partial eigenvalue assignment and partial eigenstructure assignment technique to update FE models in physical space domain are discussed by Datta [5, 10]. Carvalho [8] showed how state estimates can be used to update an FE model using optimization techniques. Minas and Inman [24] has described an iterative way powered by a non-linear optimization technique of eigenstructure assignment to update model using selective elements of eigenvectors. They also proposed an pole placement technique for the systems for which mode shapes are unknown. Several other techniques also exists in literature to update any parametric model through ESA technique [7, 9, 22, 34, 35].

ESA-based updating technique thus has been extensively used in physical space (second order) or in state space (first order) in the fields of aerospace or vehicular motion control. Andry et al. demonstrated how Moore's [25] algorithm can be used to update eigenstructure of a linear system on an airplane model. However, these types of systems do not contain any particular structure in its state matrix, whereas state space model of mechanical system has a specific structure, which can be exploited to gain more information about the system and therefore requires to be unaltered. Updating of FE models using ESA technique is thus characteristically different as some extra constraints are needed to be satisfied. In literature, these problems are mostly done in physical space, while a state space model offers greater flexibility for simultaneous updating of FEM system matrices.

Zimmerman [36, 37] has shown in his paper that it is possible to find a symmetric correction matrix to update FE model such that inherent properties of system matrices remains intact. However, symmetry and other exploitable properties of updated stiffness and damping matrix can be identified as major concern in most of these literatures. Existing methods mostly used optimization techniques with different objective functions to forcefully maintain these properties. Application of these techniques in FEM updating of real life civil infrastructure systems where assignable eigenstructure needs to be identified from noisy response signal and thus vulnerable to noise contamination is also not abundant. Therefore in this study, Moore's algorithm is employed in state space form of the system while ensuring controlled updating of the system matrices to maintain its exploitable structure by a suitable selection of input matrix. However, this approach demands that state space eigenstructure of the system needs to be identified.

Therefore the system is firstly identified through subspace identification (SSI) algorithm, and then control theory-based eigenstructure assignment (ESA) technique is employed to update the state space model of the structure in a way that the eigenstructure of its state matrix coincides with that of the state matrix identified through SSI. However, SSI technique presents few challenges in the identification procedure which needs special attention. As the measured signal contains noise, identification of the state matrix becomes inconsistent throughout the length of the time history presenting different state matrix for different segment of the signal. Added to that, for different segment of the signal SSI technique can identify the same state matrix but rotated by an unknown magnitude. This creates the requirement of standardizing the identified state matrix for each segment. An averaging step following the standardization of state matrix is adopted to minimize the influence of noise in the signal. The eigenstructure of the identified continuous time state matrix is then used to update the continuous time state matrix obtained through FE model using Moore's algorithm [25] of eigenstructure assignment.

2 Background

2.1 Discrete time stochastic subspace identification

Generally FE model updating procedure begins with an identification step. Identified system properties are then used to update the FE model. State space identification is a popular identification technique in this regard. It is always possible to transform differential equations of nth order of a dynamic system to $2n$ coupled differential equations of first order, termed as state space form of the dynamic system. Consider a dynamical system with n_1 DOF. The linear second-order differential equation of motion of the dynamic system can be written in state space form as:

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c\mathbf{x}(t) + \mathbf{B}_c\mathbf{u}(t) \tag{1}$$

where

$$\mathbf{A}_c = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}; \quad \mathbf{x}(t) = \begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix}; \quad \mathbf{B}_c = \begin{bmatrix} 0 \\ -\mathbf{M}^{-1}\mathbf{f} \end{bmatrix}; \tag{2}$$

$\mathbf{M}, \mathbf{C}, \mathbf{K}$ are the time invariant mass, damping and stiffness matrices, respectively, of $n_1 \times n_1$ order of the dynamic system. $\ddot{\mathbf{q}}(t), \dot{\mathbf{q}}(t)$ and $\mathbf{q}(t)$ are the acceleration, velocity and displacement vectors, respectively, of order $n_1 \times 1$ at the time instant t. $\mathbf{F}(t) = \mathbf{f}\mathbf{u}(t)$ is the force vector of order $n_1 \times 1$ at time instant t. $\mathbf{A}_c$ and $\mathbf{B}_c$ are state and input matrix, respectively, and subscript c represents that the equations constitute a continuous time state space model of the process.

The output or measurement equation can be written in state space form as:

$$\mathbf{y}(t) = \mathbf{C}_c\mathbf{x}(t) + \mathbf{D}_c\mathbf{u}(t) \tag{3}$$

where $\mathbf{C}_c$ is the output matrix relating states to output $\mathbf{y}(t)$ which is generally acceleration time history measured from the system and $\mathbf{D}_c$ is the direct transmission matrix relating input to output. Equations (1) and (3) describe a continuous time state space model. The discrete counterpart when the signal is sampled at intervals of δt can be expressed as:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \tag{4a}$$

$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{D}\mathbf{u}_k \tag{4b}$$

where $\mathbf{x}_k = \mathbf{x}(k\delta t)$ and $\mathbf{u}_k = \mathbf{u}(k\delta t)$ are the state vector and input vector, respectively, in discrete time state space model at the kth time step. The discrete counterparts of $\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c$ and $\mathbf{D}_c$ are:

$$\mathbf{A} = e^{\mathbf{A}_c\Delta t}; \quad \mathbf{B} = [\mathbf{A} - I]\mathbf{A}_c^{-1}\mathbf{B}_c; \quad \mathbf{C} = \mathbf{C}_c; \quad \mathbf{D} = \mathbf{D}_c; \tag{5}$$

Equation (4a) describes a deterministic system in its state space form. If the system is stochastic and the input is unknown, the input and direct transmission matrices can be taken to be zero [31]. Further, if noise is present in the input and the measured output, Eq. (4a) needs to be rewritten as:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_\mathbf{k} + \mathbf{w}_\mathbf{k} \tag{6a}$$

$$y_k = \mathbf{C}\mathbf{x}_\mathbf{k} + \mathbf{v}_\mathbf{k} \tag{6b}$$

where $\mathbf{w}_k$ and $\mathbf{v}_k$ are process and measurement noises, respectively, with covariance matrices

$$\mathbf{E}\left[\begin{pmatrix} \mathbf{w_i} \\ \mathbf{v_i} \end{pmatrix} \begin{pmatrix} \mathbf{w_j^T} & \mathbf{v_j^T} \end{pmatrix}\right] = \begin{bmatrix} \mathbf{Q} & \mathbf{S} \\ \mathbf{S^T} & \mathbf{R} \end{bmatrix} \delta_{\mathbf{ij}}$$

Equation (6) can be expressed by a forward innovation model by applying Kalman's innovation form as [20]:

$$\hat{\mathbf{x}}_{k+1} = \mathbf{A}\hat{\mathbf{x}}_k + \mathbf{K}_g\mathbf{e}_k \tag{7a}$$

$$y_k = \mathbf{C}\hat{\mathbf{x}}_k + \mathbf{e}_k \tag{7b}$$

where $\hat{\mathbf{x}}_k$ is the estimate of $\mathbf{x}_k$, $\mathbf{e}_k$ is the innovation vector and $\mathbf{K}_g$ is called Kalman gain. $\mathbf{e}_k$ is usually modeled as zero mean white noise. The system matrices $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{K_g}$ can be identified using stochastic subspace algorithm given by Overschee and Moor (See page no: 61 in the [31]) in their book titled "Subspace identification for linear systems: Theory-Implementation-Applications".

2.2 Eigenstructure assignment (ESA)

Control theory has been adopted in this paper to update the FE model using previously identified system property. Feedback is used in control to ensure stability characteristics of the system, to reduce the sensitivity of the system to modelling error, to improve the system's capability to reject disturbances and to alter the transient response of the system [3]. In this eigenstructure assignment-based model updating method concept of state feedback is adopted to assign desired eigenvalues and eigenvectors simultaneously to the system matrices. This is done by using a controller or gain matrix to update the existing system matrix such that its eigenstructure matches with the desired eigenstructure.

Consider the state space form of a dynamic system as given in the following Eq. (8). $\mathbf{D}$ matrix is ignored here considering there is no feed through in the system:

$$\mathbf{x}_{k+1} = \mathbf{A}_e\mathbf{x}_k + \mathbf{B}_e\mathbf{u}_k \tag{8a}$$

$$\mathbf{y}_k = \mathbf{C}_e\mathbf{x}_k \tag{8b}$$

Transient response of a system depends mostly on the eigenvalues and eigenvectors of the system. In order to alter the system's transient response its eigenstructure must be changed to the desired structure. In control theory-based updating an appropriate input vector sequence is selected $[\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \ldots]$ so as to place the previously identified system eigenstructure into the state matrix $\mathbf{A}_e$ at respective locations using system feedback. This process is known as eigenstructure assignment. In control theory, feedback uses external input to place this eigenstructure into the system matrix. But in the proposed approach feedback is used virtually to update the system matrix. Consider the dynamic system given in Eq. (8). The control vector $\mathbf{u}_k$ is defined as a function of the state vector as: $\mathbf{u}_k = \mathbf{K}_c\mathbf{x}_k$ where $\mathbf{K}_c$ is the gain matrix. Thus Eq. (8) can be rewritten as:

$$\mathbf{x}_{k+1} = (\mathbf{A}_e + \mathbf{B}_e\mathbf{K}_c)\mathbf{x}_k = \bar{\mathbf{A}}_e\mathbf{x}_k \tag{9}$$

This is known as full state feedback since all the states are used in the updating. Thus, if a proper gain matrix can be found which updates the system matrix in such a way that its eigenstructure matches with the desired eigenstructure, the updated system matrix will represent the system with better accuracy.

For a system described by Eq. (8) Wonham's [33] theorem enables only eigenvalue embedding to alter the system and as such does not ensure a unique solution. Moore [25] derived the necessary and sufficient conditions for a proper gain matrix $\mathbf{K}_c$ for which the updated state matrix will have the desired eigenstructure and will also be unique. This article adopts his technique of eigenstructure assignment through evaluating a gain matrix $\mathbf{K}_c$.

Moore defined $\mathbf{S}_\lambda = [\lambda\mathbf{I} - \mathbf{A}_e|\mathbf{B}_e]$ and its companion $\mathbf{R}_\lambda = [\mathbf{N}_\lambda \quad \mathbf{M}_\lambda]^T$ such that the columns of $\mathbf{R}_\lambda$ form a basis for the null space of $\mathbf{S}_\lambda$. Using partitioned basis $\mathbf{R}_\lambda$ for each λ_i the following holds true:

$$(\lambda_i\mathbf{I} - \mathbf{A}_e)\mathbf{N}_{\lambda_i} + \mathbf{B}_e\mathbf{M}_{\lambda_i} = 0 \tag{10}$$

where $\mathbf{N}_\lambda$ is a square matrix with linearly independent columns of size n and $\mathbf{N}_{\lambda^*} = \mathbf{N}_\lambda^*$, where $(*)$ represents complex conjugate. The problem at hand given by the Eq. (9) can be written as:

$$\begin{bmatrix} \lambda_i\mathbf{I} - \mathbf{A}_e & \mathbf{B}_e \end{bmatrix} \begin{bmatrix} \mathbf{v}_i \\ -\mathbf{K}_c\mathbf{v}_i \end{bmatrix} = 0 \tag{11}$$

where $\mathbf{v}_i$ s are linearly independent sets in $\mathbb{C}^n$ and $\mathbf{v}_i = \mathbf{v}_j^*$ when $\lambda_i = \lambda_j^*$. Since the columns of $\mathbf{R}_\lambda$ form a basis for the null space of $\mathbf{S}_\lambda$ for each $\lambda_i, i = 1, 2, \ldots, n$, one can conclude that $\mathbf{v}_i \in$ span of $\mathbf{N}_\lambda$, and thus:

$$\mathbf{v}_i = \mathbf{N}_{\lambda_i}\mathbf{z}_i \tag{12}$$

where z_i is some vector which relates $\mathbf{N}_{\lambda_i}$ to $\mathbf{v}_i$. Multiplying both sides of Eq. (10) by $\mathbf{z}_i$ and using Eq. (12), one gets

$$(\lambda_i I - \mathbf{A}_e)\mathbf{v}_i + \mathbf{B}_e\mathbf{M}_{\lambda_i}\mathbf{z}_i = 0 \tag{13}$$

Comparing Eqs. (11) and (13) the gain matrix $\mathbf{K}_c$ is given by:

$$\mathbf{K}_c\mathbf{v}_i = -\mathbf{M}_{\lambda_i}\mathbf{z}_i \tag{14}$$

This holds true for all $\lambda_i, i = 1, 2, \ldots, n$ and thus gain matrix $\mathbf{K}_c$ can be obtained as:

$$\mathbf{K}_c = [\mathbf{w}_1 \quad \mathbf{w}_2 \quad \cdots \quad \mathbf{w}_n][\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]^{-1} \tag{15}$$

where $\mathbf{w}_i = -\mathbf{M}_{\lambda_i}\mathbf{z}_i$. As the desired eigenstructure is a set of self conjugate eigenvalues and eigenvectors satisfying $\lambda_1 = \lambda_2^*$ and $\mathbf{v_1} = \mathbf{v_2^*}$ and so on, the above equation can be modified as described by Andry et al. [3] as:

$$\mathbf{K}_c = \begin{bmatrix} \mathbf{w}_{r1} & \mathbf{w}_{I1} & \mathbf{w}_{r3} & \mathbf{w}_{I3} & \cdots \\ & \mathbf{w}_{rn} & \mathbf{w}_{In} & & \end{bmatrix} \begin{bmatrix} \mathbf{v}_{r1} & \mathbf{v}_{I1} & \mathbf{v}_{r3} & \mathbf{v_{I3}} & \cdots \\ & \mathbf{v_{rn}} & \mathbf{v_{In}} & & \end{bmatrix}^{-1} \tag{16}$$

where subscript r and I signifies real and imaginary part of the corresponding element. The resultant gain matrix is used in the Eq. (9) to calculate the updated state matrix.

3 FEM updating using ESA technique

This section describes the method developed in this paper for updating a primary FE model using acceleration history through ESA technique. Noise in the signal is the major challenge in SSI-based identification of the real structure from its acceleration time history. Although the singular value decomposition (SVD) step inside the SSI algorithm removes the effect of noise to some extent, the remaining noise can contaminate the output of the identification algorithm. To overcome this noise problem, this paper divides the signal into a number of segments and performs system identification on all those segments separately.

Suppose two segments of the time signal are used to identify two sets of system matrices $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{K_g}$ and $\mathbf{A}'$, $\mathbf{C}'$ and $\mathbf{K}'_{\mathbf{g}}$. Since both models are describing the same linearly time invariant stochastic system, differing at most by a linear transformation of the state vector, these two sets of matrices must be related as:

$$\mathbf{A}' = \mathbf{TAT}^{-1}; \quad \mathbf{K}'_g = \mathbf{TK}_g; \quad \mathbf{C}' = \mathbf{CT}^{-1}; \tag{17}$$

where $\mathbf{T}$ is a non-singular and therefore unique transformation matrix. Although it is not possible to independently identify this transformation matrix, one can transform the state matrix $\mathbf{A}$ into a sparse matrix (the "Luenberger transformation" [32]) with non-zero elements $a'_{nj}, j = 1,\ldots,n$ in the bottom row, unit values in the first upper off diagonal position, $a'_{ii+1} = 1, i = 1,\ldots,n-1$, and all other elements being zero:

$$\mathbf{A}' = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \ldots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{1} \\ \mathbf{a}'_{\mathbf{n1}} & \mathbf{a}'_{\mathbf{n2}} & \mathbf{a}'_{\mathbf{n3}} & \ldots & \mathbf{a}'_{\mathbf{nn}} \end{bmatrix} \tag{18}$$

This transformation yields almost consistent results for each of these segments used for identification. A time averaging of this transformed state matrix gives a better estimation of the state matrix and thus noise effect can be minimized. This averaged $\mathbf{A}'$ corresponds to discrete time system and therefore requires to be transformed to continuous time domain by zero order hold (ZOH) method and then eigenvalues of the transformed state matrix are used to update the primary FEM. Although mere placing the eigenvalues will not ensure a unique solution to this problem for which the associated eigenvectors should also be assigned. Fortunately, eigenvectors are not much sensitive to small change in parameters of the FE model. Thus, if the primary FE model is selected sufficiently close to the real system, the eigenvectors of this primary model along with the identified eigenvalues can constitute the desired eigenstructure which must be matched by the FEM generated state matrix.

The ability of the ESA algorithm is then exploited to alter eigenstructure of the continuous time state transition matrix which is developed using system matrices of primary FEM. Thus $\mathbf{A_e}$ in Eq. (8) is actually continuous time state transition matrix $\mathbf{A_c}$ obtained using the primary FEM. The choice of $\mathbf{B_e}$ matrix in the ESA algorithm described above is however arbitrary as long as: (1) it does not affect controllability which can be ensured by specifying $Rank([\lambda_i\mathbf{I} - \mathbf{A_c B_e}]) \geq \mathbf{n}$, (2) after updating the upper half of the updated state matrix does not change. $\mathbf{B_e}$ is therefore chosen as:

$$\mathbf{B}_e = \begin{bmatrix} \mathbf{z}(n) & \mathbf{z}(n) \\ \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix}; \mathbf{A}_c = \begin{bmatrix} \mathbf{z}(n) & \mathbf{I}(n) \\ \mathbf{A}_{c1} & \mathbf{A}_{c2} \end{bmatrix} \tag{19}$$

$$\begin{aligned} \bar{\mathbf{A}}_c &= \mathbf{A}_c + \mathbf{B}_e \begin{bmatrix} \mathbf{k}_1 & \mathbf{k}_2 \\ \mathbf{k}_3 & \mathbf{k}_4 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{z}(n) & \mathbf{I}(n) \\ \mathbf{A}_{c1} + \mathbf{B}_1\mathbf{k}_1 + \mathbf{B}_2\mathbf{k}_3 & \mathbf{A}_{c2} + \mathbf{B}_1\mathbf{k}_2 + \mathbf{B}_2\mathbf{k}_4 \end{bmatrix} \end{aligned} \tag{20}$$

where $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4$ are the four blocks of gain matrix $\mathbf{K_c}$. $\mathbf{z(n)}$ and $\mathbf{I(n)}$ are the null and identity matrix of order n.

The benefit of this technique is that it preserves the structure of the state matrix and one can identify local changes in the stiffness or damping matrix due to the updating. Spurious weak coupling between physically non-connected DOFs are usually insignificant. Finally, any one out of the three system matrices (stiffness, mass or damping) must be assumed to have been truly modelled in this updating process, and thus can be considered to be unchanged after updating. Commonly the mass matrix is considered to be unchanged, and the updated stiffness and damping matrices ($\mathbf{K_u}$ and $\mathbf{C_u}$) are obtained from the lower left block and the lower right block of the state matrix as:

$$\begin{aligned} \mathbf{K}_u &= -\mathbf{M}(\mathbf{A}_{c1} + \mathbf{B}_1 k_1 + \mathbf{B}_2 k_3); \\ \mathbf{C}_u &= -\mathbf{M}(\mathbf{A}_{c2} + \mathbf{B}_1 k_2 + \mathbf{B}_2 k_4) \end{aligned} \tag{21}$$

4 Numerical validation

The proposed method is first validated on a numerical model of a four-story shear frame building (Fig. 1) subjected to ambient vibrations. The "real system" in this case is a lumped mass model with 1 % Rayleigh damping, for which stiffness ($\mathbf{K_0}$) and mass matrices ($\mathbf{M_0}$) are as follows:

$$\mathbf{K}_0 = \begin{bmatrix} 800 & -800 & 0 & 0 \\ -800 & 2400 & -1600 & 0 \\ 0 & -1600 & 4000 & -2400 \\ 0 & 0 & -2400 & 5600 \end{bmatrix} \mathrm{kN/m};$$

$$\mathbf{M}_0 = \begin{bmatrix} 1500 & 0 & 0 & 0 \\ 0 & 3000 & 0 & 0 \\ 0 & 0 & 3000 & 0 \\ 0 & 0 & 0 & 4500 \end{bmatrix} \mathrm{kg};$$

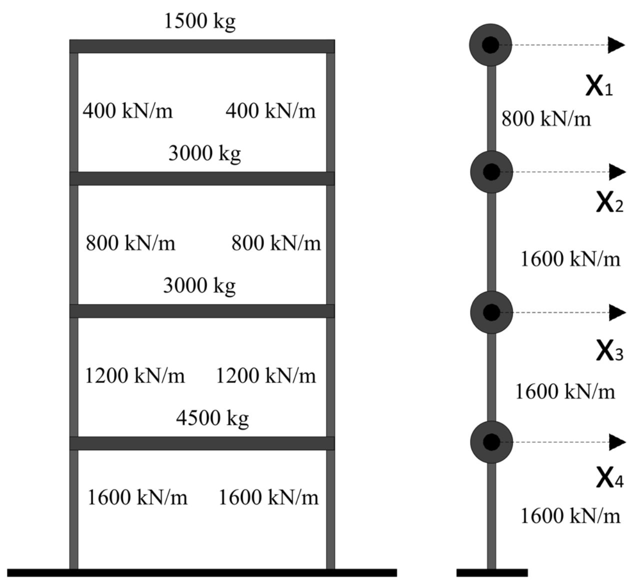


Fig. 1 Schematic diagram of four-story shear frame building and its lumped mass model

The primary model of the same system is then created with the same mass matrix, a different damping ratio (0.1 %) and a different stiffness matrix $\mathbf{K}_m$:

$$\mathbf{K}_m = \begin{bmatrix} 700 & -700 & 0 & 0 \\ -700 & 2100 & -1400 & 0 \\ 0 & -1400 & 3500 & -2100 \\ 0 & 0 & -2100 & 4900 \end{bmatrix} \mathrm{kN/m};$$

Each node of the real system is subjected to white noise excitation and the structural response is obtained using Newmark beta algorithm. The acceleration signal is recorded at all four nodes with a sampling frequency of 1000 Hz which is then contaminated with 10 % white noise for better representing the field condition. A small segment of the simulated clean signal and the contaminated signal is presented in Fig. 2. The SSI algorithm is then employed to construct state space models of the system of orders 10–100. Identification of the system dynamics in its state space domain facilitates simultaneous identification of the dynamic properties and the damping information. The eigenstructure of the identified state matrix contains this information about the system dynamics.

Eigenvalues of the state matrix identified from the acceleration signals have the form $\lambda_n = -\xi_n\omega_n \pm i\omega_n\sqrt{1-\xi_n^2}$. Figure 3 shows the stabilization plots of $\mathrm{Im}(\lambda_n)$ versus model order. Amongst the identified roots, those having physical significance stabilize over increasing model order; whereas those that are mere computational poles scatter away and do not stabilize. Each of these stable roots belongs to a particular mode of the system and

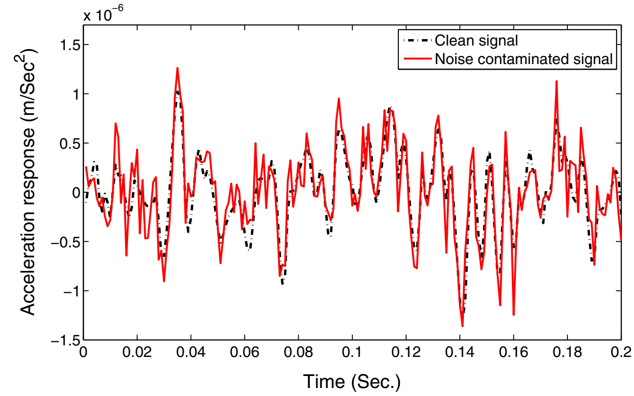


Fig. 2 Simulated clean and noise contaminated signals

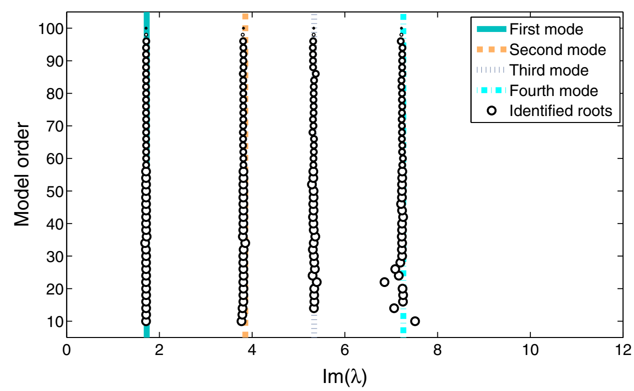


Fig. 3 Stabilization diagram of the different order state space models of the shear frame

exploiting its special structure (i.e. $\lambda_n = -\xi_n\omega_n \pm i\omega_n\sqrt{1-\xi_n^2}$) a set of natural frequency and damping ratio (ω_n and ξ_n) pair can be obtained. Thus for each identified stable root the associated mode number and $\{\omega_n, \xi_n\}$ pair are identified. For the simplistic nature of the current problem the selection of most likely $\{\omega_n, \xi_n\}$ pair from all identified $\{\omega_n, \xi_n\}$ sets is not difficult. However for complex problems the identified $\{\omega_n, \xi_n\}$ sets are usually a little scattered which may create problem in identification of most likely $\{\omega_n, \xi_n\}$ pair. Thus for complex problems for each mode all associated $\{\omega_n, \xi_n\}$ pairs are plotted in 2D histograms and the mode values of these histograms are selected as the most likely $\{\omega_n, \xi_n\}$ set. This strategy is attempted in the experimental verification problem.

Identified $\{\omega_n, \xi_n\}$ pair is subsequently used to reconstruct the state space eigenstructure which is used to update the primary FE model of the system. The updated stiffness ($\mathbf{K_{up}}$) and damping matrices ($\mathbf{C_{up}}$) are presented below.

$$\mathbf{K}_{up} = \begin{bmatrix} 796.5 & -804.3 & 0.007 & 0.008 \\ -804.3 & 2411.5 & -1614.0 & 0.002 \\ 0.007 & -1614.0 & 4018.6 & -2429.1 \\ 0.008 & 0.002 & -2429.1 & 5622.9 \end{bmatrix} \text{kN/m};$$

$$\mathbf{C}_{up} = \begin{bmatrix} 0.6546 & -0.4517 & -0.0003 & -0.0004 \\ -0.4517 & 1.7761 & -0.8733 & 0.0025 \\ -0.0003 & -0.8733 & 3.1941 & -1.0967 \\ -0.0004 & 0.0025 & -1.0967 & 4.3561 \end{bmatrix} \text{kN} - \text{m/sec};$$

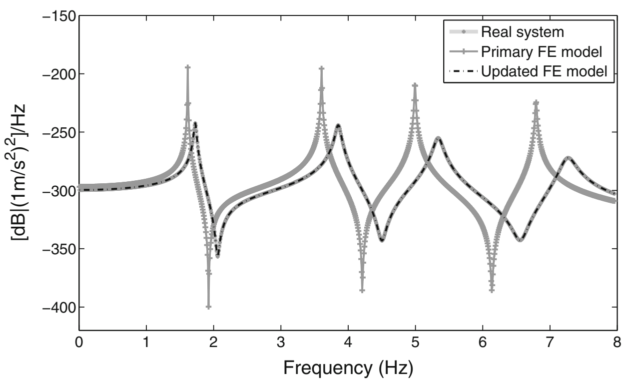


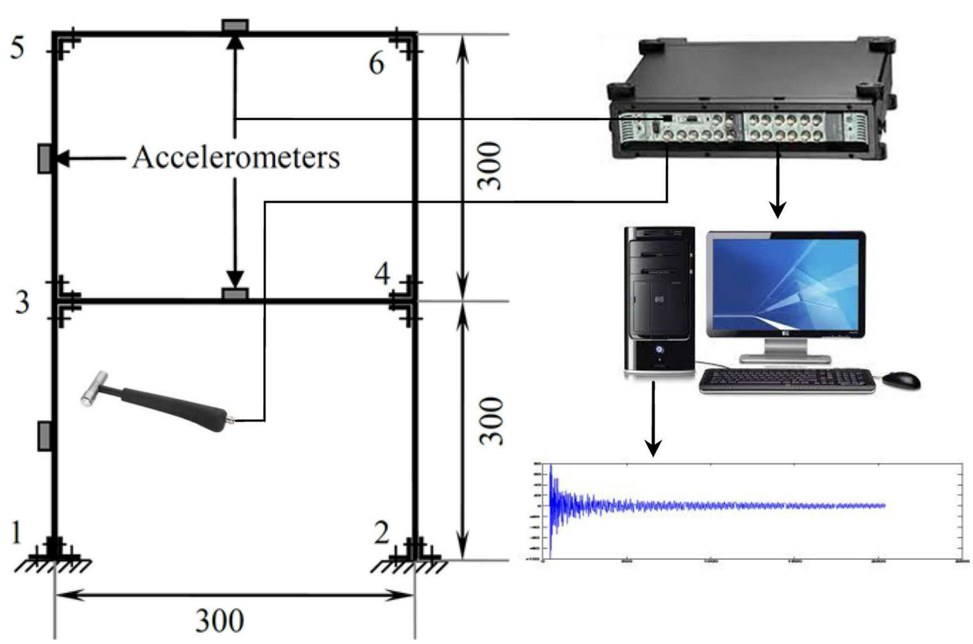
Fig. 4 Frequency response plot of real system, primary FE model and updated FE model of the shear frame building

Evidently the updated stiffness and damping matrices retain their symmetry and positive definiteness after updating. Comparing the stiffness matrices of real and updated systems one can see that the proposed method locally updates each and every element of the stiffness matrix even though global properties are used for updating. From the updated stiffness matrix, the story stiffnesses of all four stories are identified as 796.5, 1615, 2403.6 and 3219.3 kN/m, which are sufficiently close (error $<1\%$) to their corresponding actual values. The noisy measurements create spurious coupling between unconnected DOFs (e.g., elements (1, 3), (1, 4), etc., in $\mathbf{K_{up}}$) which should be ignored. The frequency response diagrams for the real system, the primary FE model and the updated FE model are presented in Fig. 4. Modal properties of the real system, the primary and updated models are also listed in Table 1. These clearly

Table 1 Natural frequencies and damping ratios for all modes of real system, primary and updated FE models

Mode number	Real system		Primary model		Updated model	
	Frequency (Hz)	Damping ratio (%)	Frequency (Hz)	Damping ratio (%)	Frequency (Hz)	Damping ratio (%)
Mode 1	1.727	1	1.616	0.1	1.721	1.104
Mode 2	3.854	1	3.605	0.1	3.832	1.159
Mode 3	5.338	1.179	4.993	0.118	5.352	1.0622
Mode 4	7.262	1.465	6.793	0.146	7.287	1.543

Fig. 5 Schematic diagram of the FE model and experimental procedure



demonstrate that the updated FE model better represents the real system.

5 Experimental validation

The proposed method is then experimentally employed on a six-member-framed structure made of aluminium bars of 30 mm × 6 mm cross section instrumented with four accelerometers of type 4503 (see Fig. 5). An impact hammer of B&K 8206-002, 2.3mV/N is used to excite the frame and acceleration signals are recorded by the 3560C B&K portable pulse acquisition front end software [27] at sampling frequency of 2048 Hz.

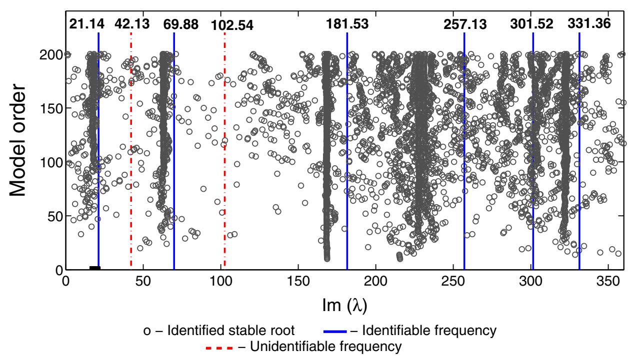
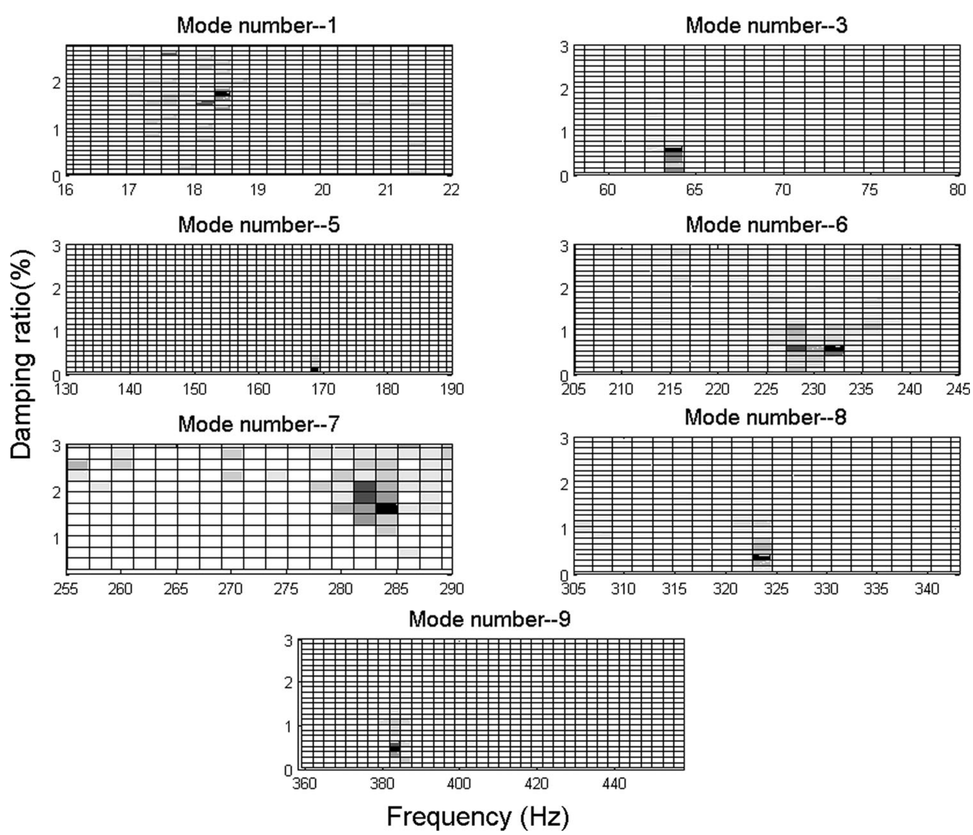


Fig. 6 Stabilization diagram of the different order state space models of the aluminium frame

As discussed previously the recorded noisy signal is put through SSI algorithm to prepare different order state space models and corresponding eigenstructures are extracted. Figure 6 shows the stabilization plots of $Im(\lambda_n)$ versus model order. All sets of identified $\{\omega_n, \xi_n\}$ pairs for each mode are plotted in 2D histograms. The 2D histograms corresponding to the first seven identified modes are shown in Fig. 7. The most likely $\{\omega_n, \xi_n\}$ pair is then obtained as the mode value (i.e., highest occurrence) of the histogram. This selected set is used to reconstruct the state space eigenvalue of the system which is subsequently used to update the primary FE model of the system, discussed next.

The primary three dimensional finite element model of the frame consists of six Euler-Bernoulli beam elements with 6 DOFs at each node. Table 2 lists the material properties used for modelling purpose. The two angles at each joint of the experimental set up ensures that the joint is rigid. Consequently each node of the FE model has been defined as moment resisting and is assigned concentrated masses to replicate the angle joints connected by bolts. The base nodes are modelled as fixed joints. Proportional 1 % Rayleigh damping is assumed for response simulation. The

Fig. 7 2D histogram plots of natural frequency and damping ratio corresponding to first seven identified eigenvalues



first ten eigenfrequencies along with corresponding mode shapes are plotted in Fig. 8. This model fails to produce the same dynamic properties as the real system and their differences are presented in Table 3. This mismatch necessitates updating the model which is undertaken by the ESA method.

Table 2 Assumed parameter values for FE modeling

Material	Aluminum
Young's modulus	70 GPa
Shear modulus	26 GPa
Density	3000 Kg/m^3
Poisson's ratio	0.346
Damping ratio	1 % Rayleigh damping

The seven eigenstructures identified above (See Fig. 7) are used to update the FE model. Table 3 lists (in the 2nd and 3rd columns) the identified frequency and damping values of these seven modes. It is important to note that these seven are not the first seven modes of the structure: the missed eigenstructures correspond to mode shapes (2nd and 4th) that are not constrained within the plane of frame structure (Fig. 8). As the experiment is performed with accelerometers that pick only in-plane acceleration signals, no out of plane modes is identified from the acceleration history. The 4th and 5th columns of Table 3 list the eigenstructure of the primary FE model. During ESA-based updating, the unidentified modes are left untouched, and only the seven identified eigenstructures are updated. The last two columns of the Table show the updated results. Updated stiffness and damping matrices are also extracted from the updated state matrix using Eq. (21) and frequency

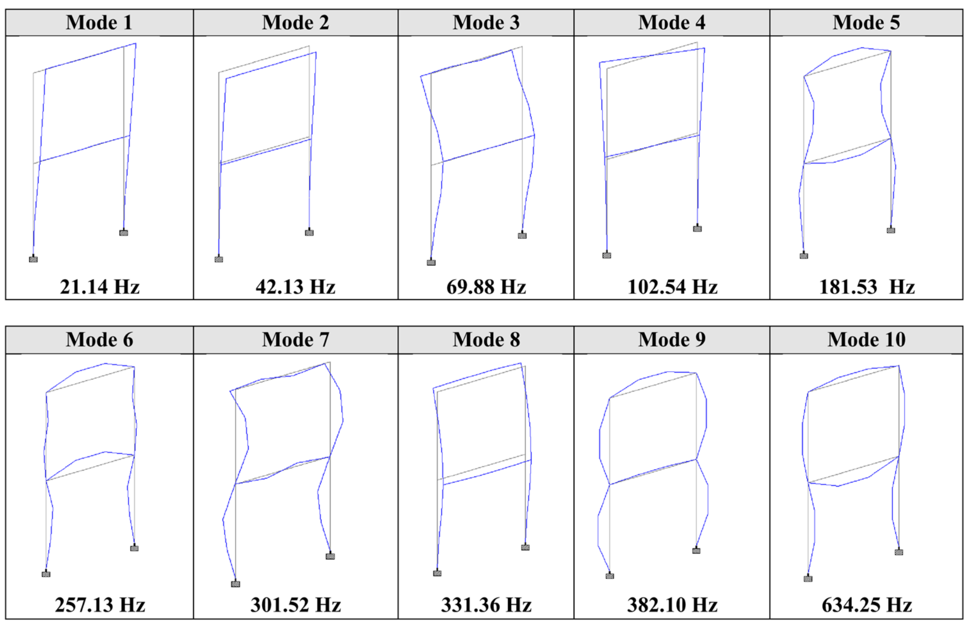


Fig. 8 Mode shape configuration corresponding to the first ten eigenfrequencies of FE model

Table 3 Natural frequencies and damping ratios related to first nine modes of real system, primary and updated FE models

Mode number	Real structure		Primary model		Updated model	
	Frequency (Hz)	Damping ratio (%)	Frequency (Hz)	Damping ratio (%)	Frequency (Hz)	Damping ratio (%)
Mode 1	18.310	1.780	21.140	1	17.831	1.971
Mode 2	Not identified	Not identified	42.138	1	42.138	1.000
Mode 3	63.180	0.670	69.879	1.305	62.427	0.606
Mode 4	Not identified	Not identified	102.541	1.758	102.541	1.758
Mode 5	168.021	0.184	181.532	2.946	166.856	0.079
Mode 6	231	0.67	257.133	4.118	227.829	0.261
Mode 7	283	1.728	301.516	4.811	281.725	1.526
Mode 8	321.9	0.43	331.363	5.279	321.710	0.287
Mode 9	381.8	0.42	382.10	6.075	381.799	0.425

response function (FRF) is obtained using these system matrices. Figure 9 shows the FRFs corresponding to the initial and updated models. As stated above, the FRFs for the second and fourth modes remain unaltered. It is clear that the updated model represents the original structure more accurately. The updated system matrices **K** and **C** also satisfy the desired exploitable properties like symmetry, definiteness, sparsity and bandedness. A negligible weak coupling between non-connected DOFs is observed. Unlike optimization techniques, the proposed method does not need to set constraints on basic properties of the system matrices. However, as updating is local in the proposed method (i.e., each element of the stiffness matrix is updated independently) physical properties such as $\mathbf{EI}, \mathbf{G}$ may be estimated in an average sense. Table 4 lists the **EI** and **G** values (mean and coefficient of variation) for all six members.

6 Conclusion

Identifying the dynamic properties of a real system and using that information to update the mathematical model of the system are challenging tasks. The challenges arise from noise present in the signal, system complexity, closely spaced modes, etc. Further, the experimental system identification has potential limitations due to limitations in number, and type and locations of sensors that can be placed on the system. A sufficiently close primary finite element model of the system can be helpful in this regard. This initial model can guide through the procedure of identification even if it fails to replicate the real system properties accurately. This primary FE model can provide rough ideas about natural frequencies, presence of closely spaced modes, etc. With this basic information, the identification of the dynamic properties from noise contaminated signals becomes less complicated.

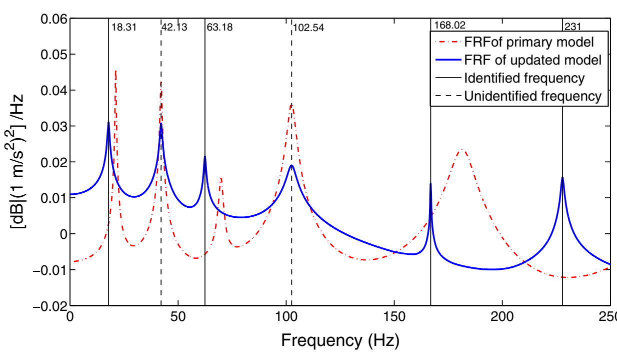


Fig. 9 Frequency response function of primary and updated FE models

In this paper, the eigenstructure assignment (ESA) method has been adopted to update the continuous time state space model of a linear time-invariant system. A subspace identification (SSI) algorithm is applied to develop the state space model and the eigenstructure of its continuous time state matrix is used to update the primary state space model developed using FEM generated system matrices. This is due to the fact that in continuous time domain the state transition matrix retains the physically explicable structure. Thus ESA algorithm given by Moore is manipulated only to use its capability of altering eigenstructure of a continuous time state matrix. A state space model is preferred over physical space model since it enables simultaneous updating of stiffness and damping matrices and also can be integrated to the control environment. The mapping of the state space back into the physical space is thus avoided.

The benefit of this method also lies in the fact that it is not iterative and hence computationally less expensive. Most FEM updating algorithms use optimization techniques in which stiffness and damping matrices are updated sequentially. Besides, optimization often suffers from problem of premature convergence and multiple optima. In ESA-based method, the state matrix gets updated just once yielding the single best possible solution to match the desired eigenstructure. Furthermore, unlike other FEM updating algorithm, it preserves the basic exploitable properties (like symmetry, sparsity, positive definiteness) of the updated stiffness and damping matrices. Connectivity between DOFs of stiffness and damping matrices is

Table 4 Approximated **EI** and **G** value obtained from updated stiffness matrix

Member	Primary model		Updated model			
	EI (N-mm^2)	**G** (GPa)	**EI** (N-mm^2) (mean)	COV (%)	**G** (GPa) (mean)	COV (%)
3	37.8×10^6	26	35.46×10^6	8.2	24.2	5.2
35	37.8×10^6	26	34.92×10^6	7.5	24.8	6.1
56	37.8×10^6	26	35.55×10^6	10.1	25.1	6.5
64	37.8×10^6	26	34.83×10^6	11.2	25.4	5.5
34	37.8×10^6	26	36.21×10^6	9.0	24.4	9.3
42	37.8×10^6	26	35.01×10^6	6.5	24.5	6.5

also maintained in most cases. However, this method demands the availability of the system matrices which most commercial FE analysis softwares do not provide due to memory constraints. Thus for large and complex systems, user-developed codes may need to be written to make system matrices available.

References

1. Allemang RJ, Brown D (1998) A unified matrix polynomial approach to modal identification. J Sound Vib 211(3):301–322
2. Alvin K, Robertson A, Reich G, Park K (2003) Structural system identification: from reality to models. Comput Struct 81(12):1149–1176
3. Andry A, Shapiro E, Chung J (1983) Eigenstructure assignment for linear systems. Aerosp Electron Syst IEEE Trans 5:711–729
4. Bagchi A (2005) Updating the mathematical model of a structure using vibration data. J Vib Control 11(12):1469–1486
5. Bai ZJ, Datta BN, Wang J (2010) Robust and minimum norm partial quadratic eigenvalue assignment in vibrating systems, a new optimization approach. Mech Syst Signal Process 24(3):766–783
6. Brincker R, Zhang L, Andersen P (2001) Modal identification of output-only systems using frequency domain decomposition. Smart Mater Struct 10(3):441
7. Cai G, Hu C, Duan G (2011) Eigenstructure assignment for linear parameter-varying systems with applications. Math Comput Model 53(5):861–870
8. Carvalho JB (2002) State estimation and finite-element model updating for vibrating systems. Northern Illinois University
9. Chu D, Chu M, Lin WW (2009) Quadratic model updating with symmetry, positive definiteness and no spill-over. SIAM J Matrix Anal Appl 31(2):546–564
10. Datta BN (2002) Finite-element model updating, eigenstructure assignment and eigenvalue embedding techniques for vibrating systems. Mech Syst Signal Process 16(1):83–96
11. Daubechies I (1992) Ten lectures on wavelets. SIAM, vol 61
12. Ewins DJ (1984) Modal testing, theory and practice, vol 79. Research studies press Letchworth
13. Friswell M, Mottershead JE (1995) Finite element model updating in structural dynamics, vol 38. Springer (1995)
14. Garrard WL, Liebst BS (1990) Design of a multivariable helicopter flight control system for handling qualities enhancement. J Am Helicopter Soc 35(4):23–30
15. Ibrahim S, Mikulcik E (1977) A method for the direct identification of vibration parameters from the free response. Shock Vib Inform Ctr Shock Vib 4:183–198
16. Imregun M, Visser W (1991) A review of model updating techniques. Shock Vib Digest 23(1):9–20
17. Juang JN (1987) Mathematical correlation of modal-parameter-identification methods via system-realization theory. Int J Anal Exp Modal Anal 2(1):1–18
18. Juang JN, Pappa RS (1986) Effects of noise on modal parameters identified by the eigensystem realization algorithm. J Guidc Control Dyn 9(3):294–303
19. Juang JN, Cooper J, Wright J (1987) An eigensystem realization algorithm using data correlations (era/dc) for modal parameter identification. Int J Anal Exp Modal Anal 4(1):294–303
20. Kalman RE et al (1960) A new approach to linear filtering and prediction problems. J Basic Eng 82(1):35–45
21. Kautsky J, Nichols NK, Van Dooren P (1985) Robust pole assignment in linear state feedback. Int J Control 41(5):1129–1155
22. Kuo YC, Lin WW, Xu S (2006) New methods for finite element model updating problems. AIAA J 44(6):1310–1316
23. Leuridan J, Brown D, Allemang R (1986) Time domain parameter identification methods for linear modal analysis, a unifying approach. J Vib Acoust Stress Reliab Des 108(1):1–8
24. Minas C, Inman D (1989) Correcting finite element model with measured modal results using eigenstructure assignment methods. State University of NewYork, Buffalo Tech. rep
25. Moore BC (1976) On the flexibility offered by state feedback in multivariable systems beyond closed loop eigenvalue assignment. Automat Control IEEE Trans 21(5):689–692
26. Peterson LD (1995) Efficient computation of the eigensystem realization algorithm. J Guid Control Dyn 18(3):395–403
27. PULSE A (2006) Solutions R 11.2, bruel and kjaer, Sound and vibration measurement A/S, Denmark
28. Skelton RE (1988) Dynamic systems control, linear systems analysis and synthesis. Wiley
29. Sobel KM, Lallman FJ, Yu W (1990) Eigenstructure assignment with gain suppression using eigenvalue andeigenvector derivatives. J Guid Control Dyn 13(6):1008–1013
30. Srinathkumar S (1988) Robust eigenvalue/eigenvector selection in linear state feedback systems, In: Decision and Control 1988 Proceedings of the 27th IEEE Conference on IEEE, pp 1303–1307
31. Van Overschee P, De Moor B (1966) Subspace identification for linear systems: theory, implementation, applications, status (published)
32. Wolovich WA (1974) Linear multivariable systems. Springer, New York
33. Wonham W (1967) On pole assignment in multi-input controllable linear systems. Automat Control IEEE Trans 12(6):660–665
34. Yuan Y, Liu H (2014) An iterative updating method for damped structural systems using symmetric eigenstructure assignment. J Comput Appl Math 256:268–277
35. Zhang J, Ouyang H, Yang J (2014) Partial eigenstructure assignment for undamped vibration systems using acceleration and displacement feedback. J Sound Vib 333(1):1–12
36. Zimmerman D, Widengren M (1989) Equivalence relations for model correction of nonproportionally damped linear systems. Dyn Control Large Struct pp 523–538
37. Zimmerman D, Widengren M (1990) Correcting finite element models using a symmetric eigenstructure assignment technique. AIAA J 28(9):1670–1676