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Probabilistic Engineering Mechanics 14 (1999) 45–54

PROBABILISTIC ENGINEERING MECHANICS

A CDM analysis of stochastic ductile damage growth and reliability

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Abstract

The accumulation of damage within a structure due to service loading or environmental conditions is a random phenomenon. Continuum damage mechanics (CDM) enables macroscopic manifestations of damage to be related to microscopic defects and discontinuities present within a material. This permits margins of safety to be assessed prior to the time at which damage becomes visible or detectable. Under fairly general thermodynamic conditions, equations of damage growth can be formulated in terms of the Helmholtz free energy. Spatial and temporal fluctuations in the state variables, caused first by the intrinsic variations in the material microstructure and second to environmental and loading conditions, are modeled by treating the Helmholtz free energy as a random process. This leads to a stochastic differential equation (SDE) of random damage growth, the solution of which describes the evolution of time-dependent random ductile damage and residual strength. Available experimental results are used to validate the CDM formulation.

Keywords: Continuum damage mechanics; Deformations; Engineering mechanics; Inelastic behavior; Probability; Reliability; Statistics; Structural engineering

1. Introduction

The state of damage within a structure and damage growth rate both are random, owing to the inherent randomness in the material microstructure, coupled with the randomness (if it exists) in the loading process and environmental fluctuations. Continuum damage mechanics (CDM) relates the effects of microstructural defects (voids, discontinuities, inhomogeneities) to quantities (stiffness, Poisson's ratio) that can be observed and measured at the macroscopic level. CDM is particularly useful in modeling accumulation of damage in a material prior to formation of a detectable defect, such as a crack. With the formation of a macroscopic defect, the essential assumption of CDM, i.e. that damage growth is a volume-wide degradation of the material microstructure, breaks down. A CDM-based approach to damage growth in the pre-localization stage can take care of its intrinsic randomness in a natural way.

In CDM, damage is represented by a state variable, D, defined as the density of defects within a cross-section of a component, amplified by their stress-raising effects [1]. In general, damage is represented by a tensor due to its directional nature [2]. When the weighted fractional loss of area on a cross-section is the same regardless of the orientation of the cross-section, then damage is *isotropic* and D becomes a scalar variable, taking values between 0 and 1. Damage is considered isotropic in the sequel. The constitutive law for a damaged material is defined by the concept of effective stress and the principle of strain equivalence [3, 4, 1]. For example, in a uniaxially loaded component, the effective stress is defined as:

$$\tilde{\sigma} = \frac{\sigma}{1 - D} \tag{1}$$

where σ is the nominal far-field stress. Applying the principle of strain equivalence, the damage variable and the fractional loss in stiffness are related by:

$$D = 1 - \frac{\tilde{E}}{E} \tag{2}$$

where $\tilde{E}$ is the elastic modulus of the damaged material, and E is the elastic modulus of a comparable undamaged material. The fractional loss of stiffness, $\tilde{E}/E$, can be measured by one of several conventional non-destructive methods, including direct tension tests, ultrasonic pulse velocity, or change in electrical resistivity, and provides the relation between damage at the microscopic and structural scales.

Existing CDM-based approaches to analysis of damage growth generally start from one of two points: (i) a kinetic equation of damage growth, or (ii) a dissipation potential function. The equations of damage accumulation that result from either approach usually lack continuity with the first principles of thermodynamics and mechanics, and introduce unknown material constants [5]. The present method addresses these shortcomings, by starting from fundamental

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0266-8920/99/$ - see front matter
PII: S0266-8920(98)00015-0

thermodynamic conditions. This approach will be validated using experimental data.

The amount of published research on stochastic CDM is small compared to the work available involving purely deterministic CDM models. CDM-based approaches are relatively new in modeling damage growth, and experimental data on random damage growth are scarce. Among the CDM-based studies of random ductile damage growth, Carmeliet and Hens [6] introduced randomness in a deterministic kinetic-equation-type formulation of damage growth in a strain-softening material by modeling the initial damage threshold and the ultimate strain as a bivariate Nataf-type random field distributed over the material. This assumption introduces complete stochastic dependence between the damage variable at two different stages of damage growth. They employed a stochastic finite element analysis for predicting the structural response of damaged components. Woo and Li [7] modeled damage growth from an assumed dissipation potential function; the stochasticity was modeled as a diffusion process with the drift term identical to the deterministic damage growth rate and the diffusion term equal to the drift term multiplied by a constant. However, no solutions were obtained for any physical damage accumulation process. In a later article, Woo and Li [8] demonstrated the statistical nature of ductile damage growth experimentally from static stress–strain tests on 45 specimens of 2024-T3 aluminum. It is apparent that stochastic CDM has not yet attained its full potential for modeling random structural damage growth in the pre-localization stage of damage accumulation.

2. Thermodynamic model of damage accumulation

Accumulation of damage is a dissipative process that is governed by the laws of thermodynamics [9]. Reviewing the deterministic formulation [5], the rate of energy dissipation for a system in diathermal contact with a heat reservoir is derived from the first and second laws of thermodynamics:

$$\Gamma \equiv -\dot{K}_{\mathrm{E}} + \dot{W} - \frac{\partial \Psi}{\partial \underline{\epsilon}} \cdot \dot{\underline{\epsilon}} - \frac{\partial \Psi}{\partial D}\dot{D} \geq 0 \tag{3}$$

where W is the work done on the system, K_{E} is its kinetic energy, and the Helmholtz free energy, $\Psi(\theta, \underline{\epsilon}, D)$, is a function of the temperature θ, the damage variable, D, and the symmetric strain tensor, $\underline{\epsilon}$. If the above system is subject to damage-causing processes that take place sufficiently close to equilibrium (in the pre-localization stage), the first variation of the free energy can be taken equal to zero [5].

The near-equilibrium system undergoes rapid and continuous transitions among its microstates. This causes random fluctuations in the state variables (which occur around their mean values), and the evolution of the free energy must be described by a stochastic process [10]. Using Eq. (3) the first variation of the free energy can be written as:

$$\delta\Psi(t) = \delta\int_{t_0}^{t}(\dot{W} - \dot{K}_{\mathrm{E}})\mathrm{d}t - \delta\int_{t_0}^{t}\Gamma\mathrm{d}t + \delta B(t) \simeq 0 \tag{4}$$

in which t_0 is the initial equilibrium state, t is an arbitrary instant of time greater than t_0 and $B(t)$ is a random process representing the random fluctuation in the free energy. In a more detailed analysis, the fluctuation $B(t)$ might be decomposed into several processes, possibly correlated, each representing contributions from individual sources such as micro-structural and environmental uncertainties. Here, however, all sources of uncertainty are assumed to be vested in $B(t)$ for simplicity.

Eq. (4) can be expressed as the difference of two integrals (with the help of Eq. (3)):

$$\begin{aligned}\delta\Psi(t) &= \delta\int_{t_0}^{t} I_1(t')\mathrm{d}t' - \delta\int_{t_0}^{t} I_2(t')\mathrm{d}t' \\ &= \int_{t_0}^{t}\delta I_1(t')\mathrm{d}t' - \int_{t_0}^{t}\delta I_2(t')\mathrm{d}t' \simeq 0\end{aligned} \tag{5}$$

where the commutability of variation and integration has been used. Terms I_1 and I_2 in Eq. (5) are defined as:

$$I_1 = \dot{W} - \dot{K}_{\mathrm{E}} + \frac{\partial\Psi}{\partial D}\dot{D} + \dot{B} \tag{6}$$

$$I_2 = \dot{W} - \dot{K}_{\mathrm{E}} - \frac{\partial\Psi}{\partial\underline{\epsilon}}\cdot\dot{\underline{\epsilon}} \tag{7}$$

where $\dot{B}(t)$ is defined in the mean-square sense. In a deformable body where damage accumulates close to thermodynamic equilibrium within and along its boundary, the second integrand, δI_2, can be shown to vanish at every instant [5]. Consequently, we are left with:

$$\int_{t_0}^{t}\delta I_1(t')\mathrm{d}t' \simeq 0. \tag{8}$$

Assuming that $\delta I_1(t') = 0$ at every instant $t' \in [t_0,t]$, the following equation can be established [5] for a deformable body undergoing isotropic damage accumulation caused by uniaxial loading:

$$\sigma_\infty + \psi_D\frac{\mathrm{d}D}{\mathrm{d}\epsilon} + s_{\mathrm{b}} = 0 \tag{9}$$

where σ_∞ is the far-field stress acting normal to the surface, ψ_{D} is the partial derivative of the free energy per unit volume, ψ, with respect to D, and $s_{\mathrm{b}} = (\partial^2 B)/(\partial\epsilon\partial V)$. While the analysis is capable of handling randomness in σ_∞, this was not done in the present study because all experimental data used in the subsequent validations were obtained for deterministic stresses. The quantity s_{b}, which has dimensions of energy per unit volume per unit strain, may be interpreted as a random fluctuation imposed on the stress field existing within the deformable body.

It is assumed that (i) s_{b} is a zero-mean process with equiprobable positive and negative values, (ii) the mean-square fluctuation in s_b is independent of strain (or time), and (iii) the rate of fluctuation in s_b is extremely rapid in comparison with the macroscopic rate of change in damage. The above assumptions are satisfied if s_b is described by the Langevin equation:

$$\frac{ds_b}{d\epsilon} = -c_1 s_b + \sqrt{c_2}\xi(\epsilon) \tag{10}$$

where $\xi(\epsilon)$ is a Gaussian white noise indexed with strain and c_1, c_2 are positive constants. The process s_b becomes stationary with variance $c_2/(2c_1)$, if c_1 is sufficiently large. Since the fluctuations in s_b are extremely rapid compared to the scale of time (or strain) of usual interest in structural mechanics, we can write the following stochastic differential equation (SDE) for damage growth [11]:

$$dD(\epsilon) = -\frac{\sigma_\infty}{\psi_D}d\epsilon - \frac{\sqrt{c_2}/c_1}{\psi_D}dW(\epsilon) \tag{11}$$

where $W(\epsilon)$ is the standard Wiener process. Note that if the strain rate is known, the random damage growth may be indexed with time, rather than with strain; here, we use strain because of the nature of the experimental data used in the subsequent verification of the CDM model. The Langevin equation assumption for the fluctuation s_b is responsible for the white noise in the damage growth rate (11); however, this is not a severe restriction on the characteristics of the noise. If the noise is colored, Eq. (11) under very general conditions can be coupled with an auxiliary equation describing the colored noise as the output of a linear filter with white noise as input.

The above formulation of damage growth admits the possibility of negative damage increments at the microscale, the probability of which depends on the relative magnitude of the drift and diffusion terms. However, in the absence of repair, the predicted increment of damage should be non-negative over a finite interval of time. Of course, this property should be verified in every situation in which the model is applied.

3. Stochastic ductile damage growth

The free energy per unit volume for uniaxial monotonic loading is:

$$\psi = \int \sigma d\epsilon - \gamma \tag{12}$$

where γ denotes the energy of formation of discontinuities per unit volume due to damage growth. The first term in Eq. (12) can be estimated by adopting the Ramberg–Osgood monotonic stress–strain relation:

$$\epsilon = \frac{\tilde{\sigma}}{E} + \left(\frac{\tilde{\sigma}}{K}\right)^M \tag{13}$$

which separates the total strain into its elastic (ϵ_e) and plastic (ϵ_p) components, and K, M are the hardening modulus and exponent respectively. The second term in Eq. (12) is estimated as:

$$\gamma = \frac{3}{4}\sigma_f D \tag{14}$$

under the assumptions that (i) the discontinuities are microscopic spheres of different sizes which do not interact with each other, (ii) the force–displacement relation is linear at the microscale, and (iii) stress amplification effects can be neglected [5]. The term σ_f is the true failure stress. Note that all parameters in Eqs. (12)–(14) can be defined from readily available handbook data, thus obviating the need for the kinetic equation or potential function approaches to damage modeling.

Applying the principle of strain equivalence, the free energy per unit volume can be written as:

$$\psi = \int \tilde{E}\epsilon_e d\epsilon_e + \int \tilde{K}\epsilon_p^{(1/M)} d\epsilon_p - \frac{3}{4}\sigma_f D \tag{15}$$

where $\tilde{E} = E(1-D)$ and $\tilde{K} = K(1-D)$ for $\epsilon_p \geq \epsilon_0$, and ϵ_0 is the threshold plastic strain for damage initiation [1]. Eq. (11) can then be written as:

$$dD(\epsilon_p) = A(\epsilon_p)(1 - D(\epsilon_p))d\epsilon_p + B(\epsilon_p)dW(\epsilon_p). \tag{16}$$

To simplify the notation, we drop the subscript p and note that ϵ denotes plastic strain in the following. The coefficients A and B in Eq. (16) are:

$$A(\epsilon) = \frac{\epsilon^{\frac{1}{M}}}{\dfrac{\epsilon^{1+\frac{1}{M}}}{1+\frac{1}{M}} + C}; \quad B(\epsilon) = \frac{(\sqrt{c_2}/c_1)/K}{\dfrac{\epsilon^{1+\frac{1}{M}}}{1+\frac{1}{M}} + C} \tag{17}$$

in which the constant:

$$C = \frac{3}{4}\frac{\sigma_f}{K} - \frac{\epsilon_0^{1+\frac{1}{M}}}{\left(1+\frac{1}{M}\right)} \tag{18}$$

incorporates the simplification $K/(2E) \sim 0$, which is valid for most aluminum and steel alloys.

Eq. (16) describes a time-dependent Ornstein–Uhlenbeck process. Since the diffusion term is independent of D, its Ito and Stratonovich solutions are identical [11]:

$$D(\epsilon) = 1 - (1 - D_0)\frac{(3/4)(\sigma_f/K)\left(1+\frac{1}{M}\right)}{\epsilon^{1+\frac{1}{M}} + \left(1+\frac{1}{M}\right)C} + \frac{(\sqrt{c_2}/c_1)/K}{\dfrac{\epsilon^{1+\frac{1}{M}}}{1+\frac{1}{M}} + C}[W(\epsilon) - W(\epsilon_0)] \tag{19}$$

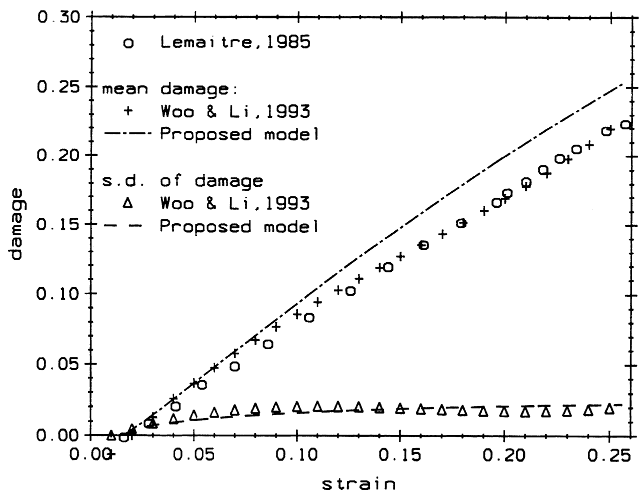

Fig. 1. Conditional mean and variance of damage when all material parameters are deterministic.

where $D_0 = D(\epsilon_0)$ = the initial damage. If the material properties $\underline{\Omega} = \{\epsilon_0, \sigma_f, K, M\}$ are considered deterministic (this assumption will be relaxed later), and D_0 is either deterministic or Gaussian, then damage is a Gaussian process. This notion is consistent with the definition of the damage variable; assuming that the random sizes of the numerous microscopic voids giving rise to damage are statistically independent of each other, damage, by its definition, can be shown to approach in distribution a normal variable when the number of defects is large.

The conditional mean of the damage process is:

$$\mu_{D|\underline{\Omega}}(\epsilon) = 1 - (1 - \mu_{D_0}) \frac{(3/4)(\sigma_f/K)}{\dfrac{\epsilon^{1+\frac{1}{M}}}{1 + \dfrac{1}{M}} + C} \tag{20}$$

which is identical to the solution found in a previous deterministic formulation [12] in which the mean initial damage, $\mu_{D_0} = 0$. The conditional variance is:

$$\sigma^2_{D|\underline{\Omega}}(\epsilon) = \sigma^2_{D_0} \left(\frac{(3/4)(\sigma_f/K)}{\dfrac{\epsilon^{1+\frac{1}{M}}}{1 + \dfrac{1}{M}} + C} \right)^2 + (\epsilon - \epsilon_0) \times \left(\frac{(\sqrt{c_2}/c_1)/K}{\dfrac{\epsilon^{1+\frac{1}{M}}}{1 + \dfrac{1}{M}} + C} \right)^2 \tag{21}$$

where $\sigma^2_{D_0}$ is the variance in the initial damage, which is equal to zero if the initial condition is deterministic. Finally, the autocovariance function of the process, for $\epsilon_2 \geq \epsilon_1 \geq \epsilon_0$, is:

$$\mathrm{cov}[D(\epsilon_1), D(\epsilon_2)|\underline{\Omega}] = \sigma^2_{D_0} \left(\frac{(3/4)(\sigma_f/K)}{\dfrac{\epsilon_1^{1+\frac{1}{M}}}{1 + \dfrac{1}{M}} + C} \right) \left(\frac{(3/4)(\sigma_f/K)}{\dfrac{\epsilon_2^{1+\frac{1}{M}}}{1 + \dfrac{1}{M}} + C} \right) + (\epsilon_1 - \epsilon_0) \left(\frac{(\sqrt{c_2}/c_1)/K}{\dfrac{\epsilon_1^{1+\frac{1}{M}}}{1 + \dfrac{1}{M}} + C} \right) \left(\frac{(\sqrt{c_2}/c_1)/K}{\dfrac{\epsilon_2^{1+\frac{1}{M}}}{1 + \dfrac{1}{M}} + C} \right) \tag{22}$$

which reduces to $\sigma^2_{D|\underline{\Omega}}(\epsilon)$ when $\epsilon = \epsilon_1 = \epsilon_2$. Ductile damage growth is clearly a non-stationary process.

4. Validation with experimental data

To validate the proposed approach, we compare our predictions of ductile damage growth in 2024-T3 aluminum with experimental results from Woo and Li [8]. Performing uniaxial load tests on 45 specimens of 2024-T3 aluminum obtained from the same batch, Woo and Li [8] obtained the means and standard deviations of $\tilde{E}$ and $\tilde{\nu}$ (damaged Poisson's ratio) at fixed intervals of ϵ (1%, 2%, ..., 25%). They also obtained the first four moments of the initial (undamaged) values of σ_y (yield stress), E and ν. They however did not compute the autocovariance functions of $\tilde{E}$ and $\tilde{\nu}$, which would have led to a more complete characterization of the random damage growth process. Interestingly, Woo and Li [8] reported a very high coefficient of variation for the threshold strain, $V_{\epsilon_0} = 1.05$.

The material parameters initially are treated deterministically in the validation, and are equated to their nominal values. The uniaxial stress–strain curve of 2024-T3 aluminum is described by the Ramberg–Osgood material parameters whose nominal values are: $E = 74\,500$ MPa, $K = 680$ MPa and $M = 5.5$. The nominal failure stress, $\sigma_f = 435$ MPa (the four preceding parameters are from Hansen and Schreyer [9]), and the nominal threshold strain, $\epsilon_0 = 0.016$ [1]. The initial damage is assumed to be zero ($\mu_{D_0} = 0$, $\sigma_{D_0} = 0$) since the experiments were performed on previously unstressed virgin specimens. The applied stress was deterministic, increasing monotonically from zero. The quantity $\sqrt{c_2}/c_1$, which is related to the ratio of the variance and the correlation length of the fluctuating quantity s_b, was selected to model the overall magnitude of the standard deviation in damage [8], and unless otherwise noted, equals 20 MPa in the following.

Fig. 1 shows the predicted $\mu^2_{D|\underline{\Omega}}(\epsilon)$ and $\sigma^2_{D|\underline{\Omega}}(\epsilon)$ for ductile damage growth. The (conditional) mean function, which is identical to the deterministic solution, agrees well with the experimental mean damage from Woo and Li [8]. It also

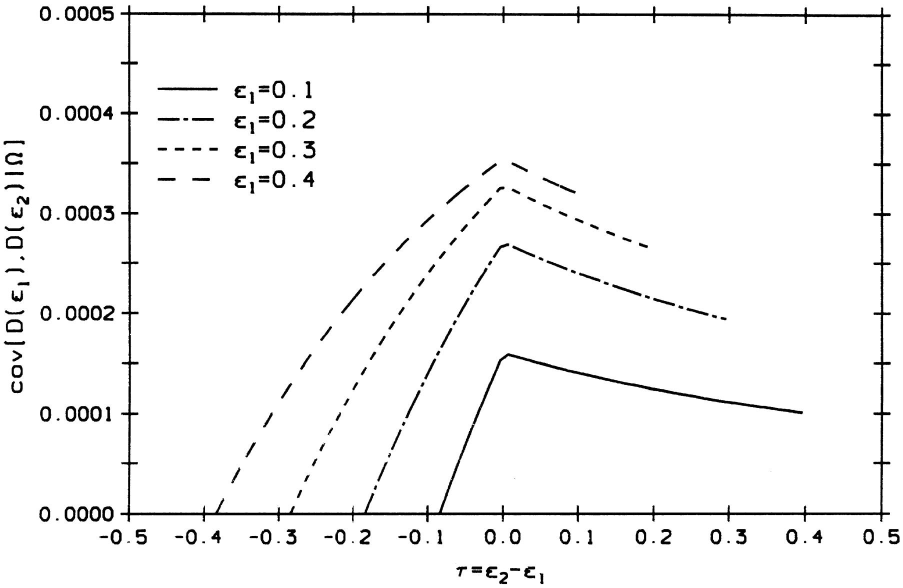

Fig. 2. Conditional autocovariance function when all material parameters are deterministic.

compares well with data reported by Lemaitre [1]; unfortunately, statistical information about Lemaitre's data is unavailable. The predicted (conditional) standard deviation in damage growth also agrees well with Woo and Li's [8] experimental results and, in particular, predicts correctly the decelerated growth in the experimental standard deviation with increasing strain.

Fig. 2 illustrates the (non-stationary) conditional covariance function of the damage process for different strain lags, $\tau = \epsilon_2$-ϵ_1, at four different values of ϵ_1. At any given value of the lag, τ, the covariance function is seen to increase with increasing ϵ_1. The non-symmetry about $\tau = 0$ arises from the non-stationary nature of stochastic damage accumulation. Experimental data needed to confirm the predicted correlation structure of random ductile damage growth in 2024-T3 aluminum were not available.

If the joint probability density of $\underline{\Omega}$ is known, then the unconditional mean and variance of the damage process

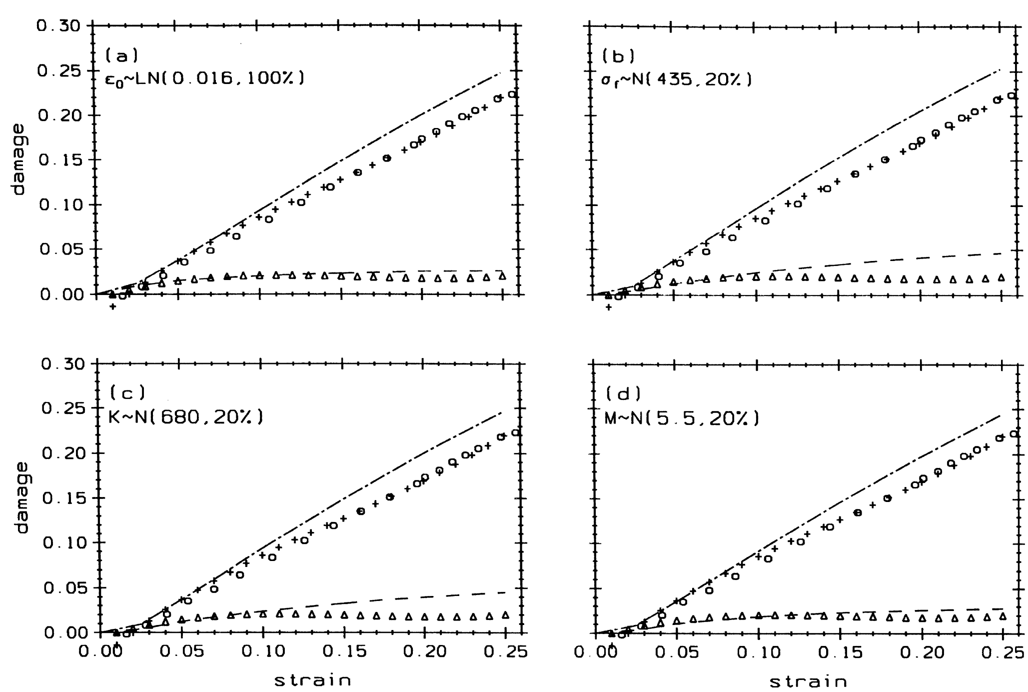

Fig. 3. Effect of randomness in material parameters and threshold strain.

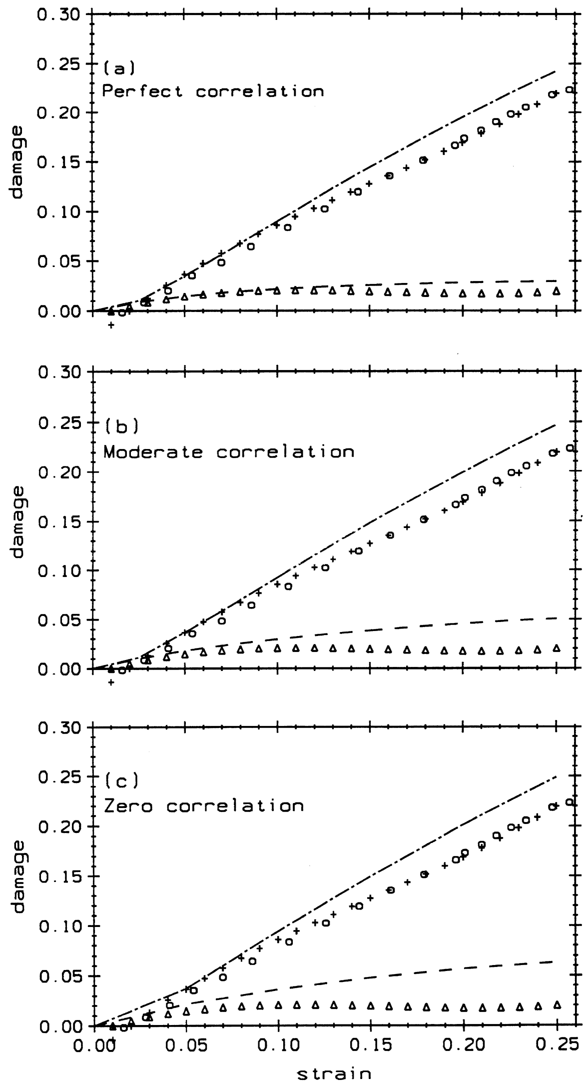

Fig. 4. Effect of correlation among the random variables.

may be obtained as:

$$\mu_D(\epsilon) = \int \dots \int \mu_{D|\underline{\Omega}}(\epsilon) f_{\underline{\Omega}}(\underline{\omega}) d\underline{\omega} \tag{23}$$

$$\sigma_D^2(\epsilon) = \int \dots \int \sigma_{D|\underline{\Omega}}^2(\epsilon) f_{\underline{\Omega}}(\underline{\omega}) d\underline{\omega} \tag{24}$$

using the theorem of total probability. $D(\epsilon)$ is generally non-Gaussian in this case and the unconditional distribution and covariance structure of $D(\epsilon)$ must be obtained numerically.

When a material parameter is treated as a random variable in the subsequent analysis, its mean is assumed equal to the nominal value. When considered as random, σ_f, K, and M are assumed to be normal variables (with coefficient of variation assumed to be 20% in each case), and ϵ_0 is assumed to be lognormal with coefficient of variation 100% (based on [8]).

Fig. 3(a)–(d) show the effect of treating ϵ_0, σ_f, K and M as random, one at a time. The noise parameter, $\sqrt{c_2}/c_1 = 20$ MPa, as before. (The plotting symbols in Fig. 3, and in Figs 4 and 5 that follow, are as defined in Fig. 1.) It is observed that the mean damage is not appreciably different in any of the four cases. The standard deviation of damage is quite insensitive to the randomness in ϵ_0 or M, but increases when σ_f or K is considered random. These findings are consistent with the sensitivity analyses performed on the deterministic version of the present model [5].

Fig. 4(a)–(c) show the effect of treating all four variables as random but with varying degrees of correlation. When the random variables are 'perfectly correlated' (Fig. 4(a)), the correlation matrix of $\ln(\epsilon_0)$, σ_f, K, M is the identity matrix; and when 'moderately correlated' (Fig. 4(b)), the off-diagonal terms of the correlation matrix are all taken to be 0.5. The noise parameter in the model is $\sqrt{c_2}/c_1 = 20$ MPa and the experimental data points are the same as before. These Figures may be compared with Fig. 1 where the parameters are all deterministic. It is observed that treating the parameters as random and varying the correlation among them has almost no effect on the mean damage. However, the standard deviation of $D(\epsilon)$ is significantly affected by randomness in all variables, and is highest when the random variables are considered to be mutually statistically independent (Fig. 4(c)). It may therefore be useful to know the correlation structure of the initial condition and material parameters to obtain accurate predictions of damage accumulation. Considering them as independent may over-estimate the scatter in damage by a factor of 2.

Fig. 5(a)–(c) show the effect of the noise intensity, $\sqrt{c_2}/c_1$, on the damage growth process in the presence of moderate correlation ($\rho = 0.5$) among the random variables $\ln(\epsilon_0)$, σ_f, K, M. Several values of the unknown noise intensity $\sqrt{c_2}/c_1$ were tried, and their effects on the sample paths of damage growth were observed while holding all parameters as deterministic. Values above 40 MPa produced occasional but significant negative damage increments, and consequently the noise parameter was restricted to below 40 MPa. In particular, Fig. 5(a) pertains to the case when the noise is entirely absent. These Figures (Fig. 5(a)–(c)) may be viewed together with Fig. 4(b) where $\sqrt{c_2}/c_1 = 20$ MPa and correlation is moderate. It is observed that the intensity of the noise has no effect on the mean of the process (as predicted by Eq. (20)), but has a significant effect on the standard deviation of damage.

Excellent agreement with experimental results was achieved with several combinations of the noise intensity, the correlation coefficients and the marginal distributions, suggesting that with proper parameter identification, this method can reproduce experimentally observed stochastic damage accumulation. Additional experimental data and validation studies clearly are necessary to take full advantage of the method for condition assessment of structural components and systems.

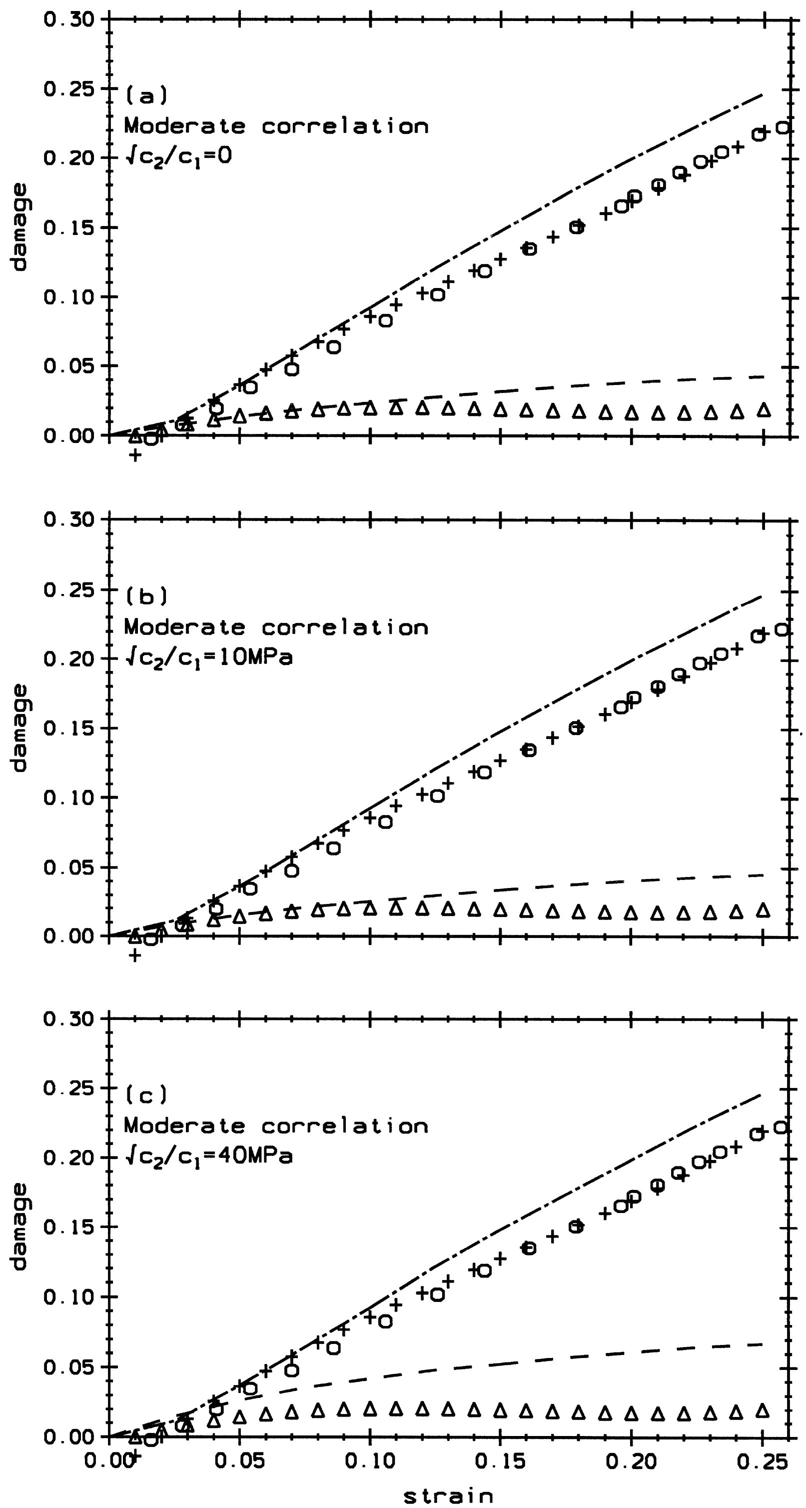

Fig. 5. Effect of noise intensity in the presence of moderate correlation.

5. Limit states and reliability in ductile deformation

Failure occurs when damage reaches the critical value D_c, which corresponds to localization in the damage growth process. In other words, 'failure' in CDM implies the breakdown of the continuum assumption, which occurs with the formation of the first macroscopic defect. This definition of failure enables damage mechanics to predict structural deterioration in the so-called initiation phase, unlike many existing methods (for example, the Paris law in fatigue) which require a measurable flaw to be useful. The critical damage is often considered to be a material property in the literature

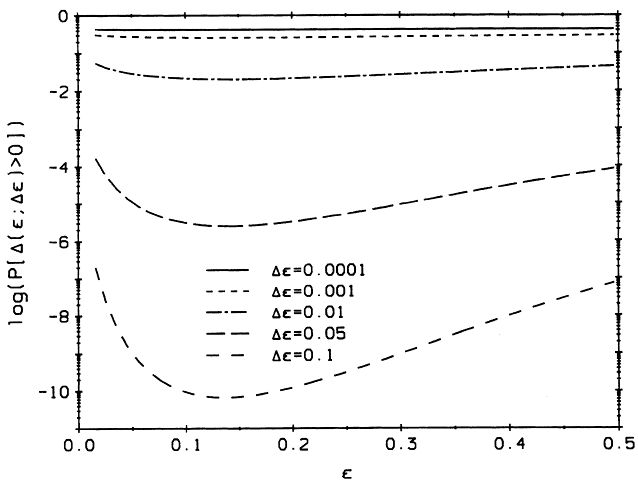

Fig. 6. Probability of a negative damage increment.

(e.g. [13]), with a value that usually ranges between 0.15 and 0.85 for engineering alloys [14]. In a stochastic formulation, D_c should be treated as a random variable.

The cumulative failure probability (CFP) at strain ϵ, $F_{\epsilon_f}(\epsilon)$, is:

$$F_{\epsilon_f}(\epsilon) = 1 - \mathrm{P}[D(\epsilon') \le D_c; \forall \epsilon' \in [0, \epsilon]] \tag{25}$$

where ϵ_f is the failure strain. Mathematically, this is a first passage problem.

The probability of a negative damage increment in the present formulation depends on the drift and diffusion terms in Eq. (16) and also on the time (or strain) interval of observation. If the drift in damage growth is large compared to its diffusion, then the growth rate is almost always positive. The increment in damage, $\Delta D(\epsilon;\Delta\epsilon)$, over an interval of plastic strain, $[\epsilon,\epsilon + \Delta\epsilon]$, is a random quantity. Its conditional mean and variance are:

$$\mathrm{E}[\Delta D(\epsilon;\Delta\epsilon)|\underline{\Omega}] = A(\epsilon)(1 - \mu_{D|\underline{\Omega}}(\epsilon))\Delta\epsilon \tag{26}$$

$$\mathrm{var}[\Delta D(\epsilon;\Delta\epsilon)|\underline{\Omega}] = (A(\epsilon))^2\Delta\epsilon^2(\sigma_{D|\underline{\Omega}}(\epsilon))^2 + (B(\epsilon))^2\Delta\epsilon. \tag{27}$$

The conditional probability of a negative damage increment over the range $\Delta\epsilon$:

$$\mathrm{P}[\Delta D(\epsilon;\Delta\epsilon) \le 0|\underline{\Omega}] = \Phi\left[\frac{-A(1-\mu_{D|\underline{\Omega}})}{\sqrt{A^2\sigma^2_{D|\underline{\Omega}} + B^2/(\Delta\epsilon)}}\right] \tag{28}$$

in which $\Phi(\cdot)$ is the standard normal cumulative distribution function (CDF), is a function of the position as well as the length of the interval. Fig. 6 shows the probability of a negative damage increment over different interval sizes as

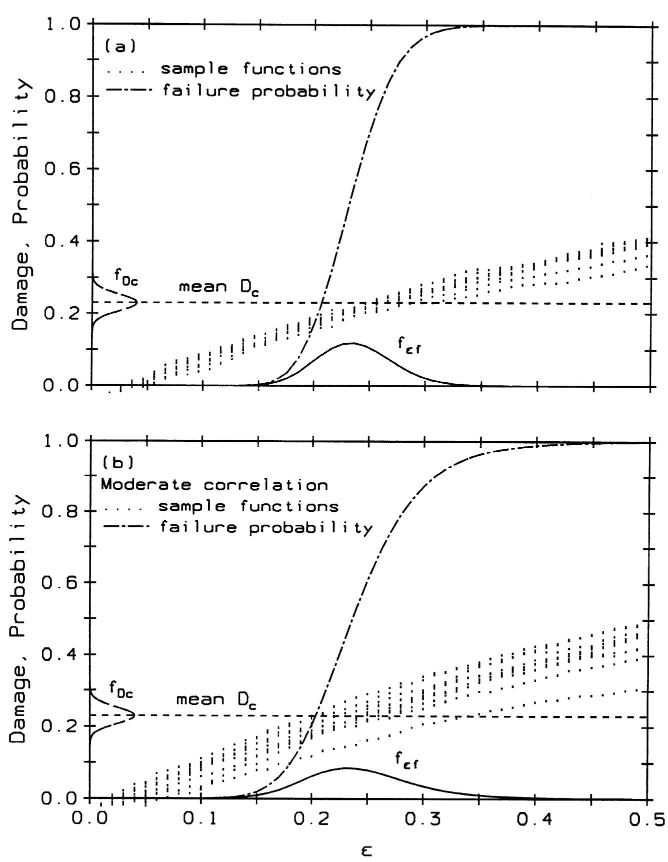

Fig. 7. Failure probability and sample paths of damage (a) only D_c random (b) all five parameters random.

a function of strain for 2024-T3 aluminum with the same material properties as before. As the interval size, $\Delta\epsilon$, approaches zero, the instantaneous growth rate approaches a 50% probability of attaining negative values, caused by the special nature of the white noise (i.e., infinite variance). However, as $\Delta\epsilon$ increases, the probability rapidly falls off to negligible quantities ($\sim 10^{-10}$ for $\Delta\epsilon = 0.1$). Under this condition, the sample paths of $D(\epsilon)$ which cross D_c (from below) for the first time at $\epsilon_1 < \epsilon$, may be expected to stay above that barrier at ϵ. This becomes more and more likely the larger the interval $(\epsilon - \epsilon_1)$ gets. In such cases, the CFP can be simplified as the CDF of the damage function evaluated at the critical damage:

$$F_{\epsilon_f}(\epsilon) \simeq 1 - \mathrm{P}[D(\epsilon) \leq D_c]. \tag{29}$$

Fixing the vector of initial condition and the material parameters, $\underline{\Omega}_1 = \{D_0, \epsilon_0, \sigma_f, K, M, D_c\}$, the cumulative failure probability can be written as:

$$F_{\epsilon_f|\underline{\Omega}_1}(\epsilon) = 1 - \Phi\left(\frac{D_c - \mu_{D|\underline{\Omega}_1}(\epsilon)}{\sigma_{D|\underline{\Omega}_1}(\epsilon)}\right) \tag{30}$$

where $\mu_{D|\underline{\Omega}_1}(\epsilon)$ and $\sigma_{D|\underline{\Omega}_1}(\epsilon)$ may be obtained by setting $\mu_{D_0} = D_0$ and $\sigma_{D_0} = 0$ in Eq. (20) and Eq. (21) respectively. The theorem of total probability may be used to remove the conditioning on $\underline{\Omega}_1$ if their joint probability density is known.

Fig. 7(a)–(b) illustrate the CFP (25) for 2024-T3 aluminum, showing that the failure strain, ϵ_f, now becomes a random variable. In Fig. 7(a), only D_c is treated as random (a normal random variable, with mean 0.23 equal to the deterministic value from Lemaitre [1], and a coefficient of variation 0.10 based on Woo and Li [8]) while the other parameters (ϵ_0, σ_f, K, M) are held constant at their nominal values. In Fig. 7(b), all five parameters are considered random (with marginal distributions and statistics the same as before) with moderate correlation ($\rho = 0.5$) between each pair. The noise intensity is 20 MPa in both Figures. To give a visual sense of the scatter in the damage accumulation process, a few sample functions of $D(\epsilon)$ (selected at random) are shown in the Figures; these sample functions were obtained numerically from Eq. (16) using an interval size $\Delta\epsilon = 0.01$. No sample function returns to the safe region once it has exited that region, reinforcing the assumption of non-negative damage growth. The scatter in the sample functions is greater in Fig. 7(b) than in Fig. 7(a), as may be expected intuitively.

The relation between D and ϵ is non-linear, since it involves the Wiener process appearing in the SDE of damage growth, and cannot be inverted explicitly. The CDF of ϵ_f is obtained numerically (from Eq. (16)) using step size $\Delta\epsilon = 0.01$, following which its probability density function is obtained by numerical differentiation. The mean and standard deviation of ϵ_f are found to be, respectively, 0.237 and 0.034 in the first case (Fig. 7(a)) and 0.247 and 0.052 in the second case (Fig. 7(b)), values which are of the same order as those generally observed for engineering metals (e.g., [15]). As may be expected, the scatter in ϵ_f increases when all the parameters are considered as random variables.

6. Conclusion

Starting from first principles of thermodynamics and recognizing the intrinsic energy fluctuations in matter, a stochastic differential equation of ductile damage growth in a deformable body is obtained. The proposed approach deals with two different aspects of randomness: one that pertains to the initial conditions and material parameters, and the other that is associated with the instantaneous growth rate of the process (in the form of a stochastic noise). A knowledge of both sources of randomness is required for a satisfactory description of stochastic damage growth. Dependence among the material parameters and their correlation structure were found to play a significant role in the scatter of the damage growth process. Finally a scheme to perform reliability analyses was outlined. Agreement with available experimental results by other researchers was good.

The proposed method also has yielded encouraging results in predicting damage growth and random crack initiation in fatigue loading and for high-temperature creep deformation [5]. Additional experimental data to further validate the model would be desirable to enhance its use as a tool for condition assessment and service life prediction of structural components and systems.

Acknowledgements

This article elaborates upon an extended abstract [10] that was presented at the 7th ASCE Specialty Conference of Probabilistic Mechanics, Worcester, MA, USA, August 1996. Support for the research, provided, in part, by the Lockheed–Martin Energy Research Corporation under Award 19X-SP638V, is gratefully acknowledged.

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