CONTINUUM DAMAGE MECHANICS-BASED MODEL OF STOCHASTIC DAMAGE GROWTH

By Baidurya Bhattacharya[1] and Bruce Ellingwood[2]

ABSTRACT: Structural damage accumulation is an intrinsically random phenomenon. Continuum damage mechanics seeks to express the aggregate effect of microscopic defects present within a material in terms of macroscopically defined quantities; this makes continuum damage mechanics well-suited to deal with random damage growth in the prelocalization stage. Growth of damage is a thermodynamically irreversible process where the evolution of the Helmholtz free energy is described by a random process. Under fairly general thermodynamic conditions, a set of stochastic differential equations are derived for random isotropic damage growth prior to the onset of localization. The notion that the current state of damage encapsulates the history of the entire process imparts a Markovian characteristic to the damage growth process. The stochastic differential equations are solved to assess damage growth and reliability for uniaxial ductile deformation, high-temperature creep, and fatigue cycling. The models are validated with available experimental results.

INTRODUCTION

Structural damage growth is a random phenomenon (Woo and Li 1993; Harlow and Delph 1995; Qian et al. 1996), due first to the intrinsic variations in the material microstructure, and second to the fluctuations in the loading process and the environment. Continuum damage mechanics (CDM) deals with the effect of microstructural defects (voids, discontinuities, and inhomogeneities) present within a material volume on properties that can be measured at the macroscopic level, such as stiffness and Poisson's ratio. CDM is particularly suited for modeling accumulation of damage in a material prior to formation of a detectable defect (localization). For many structures and loading envelopes, the prelocalization phase is a substantial portion of their service lives. A CDM-based approach to overall damage growth in the prelocalization stage can accommodate its intrinsic randomness in a natural way and enhance service life predictions of degrading structural systems.

The existing empirical models of damage growth (and their "randomized" counterparts) are usually either a posteriori in nature (such as Miner's law in fatigue) or require the existence of a measurable flaw (such as Paris's law in fracture mechanics). Stochastic models that are not related to CDM, but are based on other state-variable-type "damage" concepts, have been proposed in the literature [for example, by Diao (1995), who equated damage with the cumulative failure probability at a given time, and Krajcinovic and Silva (1982), Breysse (1990), and Kandarpa et al. (1996), who defined damage as the fraction of springs fractured in an idealized material volume]. Such models usually lack the sound thermodynamical and micromechanical bases of a CDM-based approach.

Basic Concepts of CDM

CDM defines damage as the density of defects/discontinuities on a cross section in a given orientation, amplified by their stress-raising effects (Lemaitre 1985). In general, damage is represented by tensors due to its directional nature (Krajcinovic 1984). When the weighted fractional loss of area on a cross section is the same regardless of the orientation of the cross section, then damage is isotropic and is described by a scalar variable D taking values between 0 and 1. Damage is considered to be isotropic in this paper. The concept of effective stress, along with the principle of strain equivalence (Lemaitre 1985; Kachanov 1986; Chaboche 1988), may be used to derive the constitutive law for a damaged material. For example, in a uniaxially loaded component, the effective stress is defined as

$$\tilde{\sigma} = \frac{\sigma}{1 - D} \tag{1}$$

where σ = nominal stress. Applying the principle of strain equivalence, the damage variable may be related to the fractional loss in stiffness

$$D = 1 - \frac{\tilde{E}}{E} \tag{2}$$

where $\tilde{E}$ = elastic modulus of the damaged material; and E = elastic modulus of a comparable undamaged material. Eq. (2) provides a means for measuring the extent of damage in a structural component experimentally by one of several conventional nondestructive methods, including direct tension tests, ultrasonic pulse velocity, measurement of electrical resistivity, etc. (Lemaitre 1992).

Failure occurs when the damage variable equals the critical damage D_c. In the context of CDM, failure does not necessarily correspond to rupture but rather to the formation of a macroscopic defect—the state at which an essential premise of CDM, namely that damage growth is a volume-wide degradation of the material microstructure—ceases to be applicable (Chaboche 1988). The critical damage is postulated to be a (temperature-dependent) material property [e.g., Chow and Wei (1991)], and its value usually ranges between 0.15 and 0.85 for engineering alloys (Lemaitre 1992). In a stochastic formulation, it may be more appropriate to treat D_c as a random variable.

Existing CDM-Based Models of Random Damage Growth

The existing CDM-based formulations of damage growth start with one of two approaches: (1) A kinetic equation of damage growth [e.g., Kachanov (1986)]; or (2) a dissipation potential model [e.g., Hansen and Schreyer (1994)]. These models usually lack continuity with the first principles of thermodynamics and mechanics and introduce unknown material constants, as reviewed in Krajcinovic and Mastilovic (1995) and Bhattacharya (1997). The method presented in this paper addresses these shortcomings to a substantial degree, as will

[1]Postdoctoral Fellow, Dept. of Civ. Engrg., Johns Hopkins Univ., Baltimore, MD 21218.

[2]Prof., Dept. of Civ. Engrg., Johns Hopkins Univ., Baltimore, MD.

Note. Associate Editor: Ross B. Corotis. Discussion open until February 1, 1999. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on May 5, 1997. This paper is part of the ***Journal of Engineering Mechanics***, Vol. 124, No. 9, September, 1998. ©ASCE, ISSN 0733-9399/98/0009-1000–1009/$8.00 + $.50 per page. Paper No. 15696.

be shown subsequently. The amount of published research on stochastic CDM is small compared to the work available involving purely deterministic models, presumably due to the relative novelty of the field and the scarcity of experimental data on CDM-related random damage growth.

Carmeliet and Hens (1994) used a nonrandom kinetic equation of damage growth and introduced randomness in the formulation by modeling the initial damage threshold and the ultimate strain as a bivariate Nataf-type random field distributed over the material. This approach introduces complete stochastic dependence between the damage variable at two different stages of damage growth. A stochastic finite-element analysis was employed for predicting structural responses of damaged components. A similar finite-element method and kinetic equation-based approach was adopted for high-temperature fatigue crack growth by Qian et al. (1996).

Woo and Li (1992) adopted a dissipation potential approach and modeled damage growth as a diffusion process with the drift term identical to the deterministic damage growth rate and the diffusion term equal to the drift term multiplied by a constant. However, Woo and Li did not solve this stochastic differential equation (SDE) (numerically or in closed-form) for any physical damage accumulation process. In a subsequent paper, Woo and Li (1993) demonstrated the statistical nature of ductile damage growth experimentally from static stress-strain tests on 45 specimens of 2024-T3 Aluminum.

Harlow and Delph (1995) adopted the familiar power law model for steady-state creep strain rate along with the Kachanov model (1986) of creep damage growth and assumed the parameters of the two models to be jointly distributed random variables. However, the existence of noise in the damage growth process was not considered. At the current state of the art, stochastic CDM has not yet attained its full potential for modeling random structural damage growth.

THERMODYNAMIC MODEL OF DAMAGE ACCUMULATION

Consider a deformable body R (defined by the closed boundary ∂R) in diathermal contact with a heat reservoir at constant temperature θ and subject to damage causing processes prior to the onset of localization. The rate of heat flow into R from the reservoir is $\dot{Q}$, and the rate of increase in the entropy of R occurs at the rate of $\dot{S}$ (the superscript dot represents total derivative with regard to time). By the second law of thermodynamics

$$\dot{S} - \frac{\dot{Q}}{\theta} \geq 0 \tag{3}$$

The Helmholtz free energy, $\Psi(\theta, \underline{\varepsilon}, D)$, defined as $\Psi = U - \theta S$, where U is the internal energy of R, is a function of the temperature, the symmetric strain tensor $\varepsilon_{ij} = 1/2(u_{i,j} + u_{j,i})$ and the damage variable. Combining (3) with the first law of thermodynamics (which states, $\dot{K}_E + \dot{U} = \dot{Q} + \dot{W}$, where W is the work done on R, and K_E is the kinetic energy of R), the overall rate of dissipation Γ can be written in terms of the evolution of the Helmholtz free energy as

$$\Gamma \equiv -\dot{K}_E + \dot{W} - \frac{\partial \Psi}{\partial \underline{\varepsilon}} \cdot \dot{\underline{\varepsilon}} - \frac{\partial \Psi}{\partial D} \dot{D} \geq 0 \tag{4}$$

The increase in the Helmholtz free energy gives the upper limit to the work in a process between two equilibrium states at the same temperature (Sears and Salinger 1975)

$$d\Psi = dW_{nk} - dW_d, \quad dW_d \geq 0 \tag{5}$$

where W_{nk} = nonkinetic work; and W_d = energy dissipation.

Randomness in damage growth from rapid and continuous transitions and interactions in the microstates of the system R (Callen 1985; Ostoja-Starzewski 1989) suggests that the Helmholtz free energy should be described as a stochastic process (Bhattacharya and Ellingwood 1996)

$$\Psi(t) = \Psi(t_0) + \int_{t_0}^{t} (\dot{W} - \dot{K}_E)\, dt' - \int_{t_0}^{t} \Gamma\, dt' + \int_{t_0}^{t} \dot{B}(t')\, dt' \tag{6}$$

where $B(t)$ = stochastic process representing the random fluctuation in the free energy; and $\dot{B}(t)$ is its derivative in the mean-square sense. The process $B(t)$ accounts for randomness in material microstructure, void interactions, thermal fluctuations, and environmental effects. Modeling errors can also be included in $B(t)$. While conceptually these effects each constitute a random process (with, possibly, significant correlation among them), it is difficult to separate these effects at the current state of knowledge, and for simplicity, all sources of uncertainty are vested in $B(t)$. Moreover, spatial fluctuations in the free energy at a given instant are neglected as they are assumed small in a nominally homogeneous material undergoing isotropic damage accumulation prior to localization.

It is assumed that the initial state (at time t_0) is one of thermodynamic equilibrium, and damage accumulation, though irreversible, occurs sufficiently close to equilibrium in the prelocalization stage. Under these assumptions, the first variation in $\Psi(t)$, which is generally nonzero for a system yet to achieve equilibrium, may be assumed to vanish to a degree sufficient for engineering purposes

$$\delta\Psi(t) = \delta\Psi(\theta, \underline{\varepsilon}, D, B, \delta\dot{\theta}, \delta\dot{\underline{\varepsilon}}, \delta\dot{D}, \delta\dot{B}, \ldots; t') \simeq 0, \quad t' \in [t_0, t] \tag{7}$$

The validity of this assumption will be tested subsequently with experimental data. Eq. (7) can be expressed as the difference of two integrals [incorporating (4)]

$$\delta\Psi(t) = \int_{t_0}^{t} \delta I_1(t')\, dt' - \int_{t_0}^{t} \delta I_2(t')\, dt' \simeq 0 \tag{8}$$

noting that the variation and integration operators commute. The terms I_1 and I_2 in (8) are

$$I_1 = \dot{W} - \dot{K}_E + \frac{\partial \Psi}{\partial D} \dot{D} + \dot{B} \tag{9}$$

$$I_2 = \dot{W} - \dot{K}_E - \frac{\partial \Psi}{\partial \varepsilon_{ij}} \dot{\varepsilon}_{ij} \tag{10}$$

Suppose the boundary ∂R is composed of two mutually exclusive sets: (1) The free surface ∂R_1; and (2) ∂R_2 where displacements are specified. Applying a set of variations $\partial \dot{u}_i$ to the displacement field (consistent with the boundary condition $\delta\dot{u}_i(t') = 0$, on $\partial R_2\ \forall t' \in [t_0, t]$), which is small enough not to alter the instantaneous force, acceleration, and stress distributions in R, and accounting for the symmetry of the stress and strain tensors, the second integrand δI_2 can be written as

$$\delta I_2 = \int_R F_i \delta\dot{u}_i\, dV + \int_{\partial R} T_i \delta\dot{u}_i\, d\eta - \int_R \rho a_i \delta\dot{u}_i\, dV - \int_R \sigma_{ij} \delta\dot{u}_{i,j}\, dV \tag{11}$$

where F_i and T_i ($i = 1, 2, 3$) = body forces (on R) and surface forces (on ∂R_1), respectively; a_i and $\dot{u}_i$ = acceleration and velocity, respectively; ρ = mass density; the symmetric stress tensor $\sigma_{ij} = \partial\psi/\partial\varepsilon_{ij}$, where ψ is the Helmholtz free energy per unit volume. Applying Green's theorem and integrating by parts

$$\delta I_2 = \int_R (F_i + \sigma_{ij,j} - \rho a_i)\delta\dot{u}_i\, dV + \int_{\partial R_1} (T_i - \sigma_{ij} n_j)\delta\dot{u}_i\, d\eta \tag{12}$$

where n_j = outward unit normal on the elemental surface area $d\eta$. The integrands in parentheses in (12) are each equal to zero as they constitute equilibrium equations of a damaged body (Krajcinovic and Sumarac 1987). Thus the second term in (8) vanishes. Hence, the first term in (8) must also vanish.

Assume that $\delta I_1(t')$ vanishes at every instant $t' \in [t_0, t]$ and apply a set of variations $\delta\dot{u}_i$ obeying the same boundary conditions as in (11), which are small enough not to alter the instantaneous force, acceleration, and strain distributions and which do not affect the rate of change in ψ with regard to damage growth. Proceeding in a manner similar to that used to derive (12), the following is obtained:

$$\delta I_1 = \int_R [F_i - (\psi_D D'_{ij} + s_{b_{ij}})_J - \rho a_i]\delta\dot{u}_i \, dV + \int_{\partial R_1} [T_i + (\psi_D D'_{ij} + s_{b_{ij}})n_j]\delta\dot{u}_i \, d\eta \simeq 0 \tag{13}$$

where $\psi_D = \partial\psi/\partial D$, $D'_{ij} = \partial D/\partial\varepsilon_{ij}$, and $s_{b_{ij}} = \partial b/\partial\varepsilon_{ij}$, where $b = \partial B/\partial V$ is the energy fluctuation per unit volume. Choosing $\delta\dot{u}_i$, which vanish on the free surface ∂R_1, the following can be written:

$$F_i - \rho a_i - (\psi_D D'_{ij} + s_{b_{ij}})_J = 0 \quad \text{on} \quad R \tag{14}$$

Now a set of nonzero variations are chosen on the free surface, and the following are obtained:

$$T_i + (\psi_D D'_{ij} + s_{b_{ij}})n_j = 0 \quad \text{on} \quad \partial R_1 \tag{15}$$

Eq. (15) represents a coupled set of partial differential equations, which may be difficult to solve for a body subjected to multiaxial straining. However, an SDE can be derived for uniaxial straining that is amenable to closed-form solutions for different modes of damage accumulation. The fact that material properties and random damage growth data are mainly available for uniaxial loading conditions makes this SDE useful for testing the validity of the approach for modeling random structural damage accumulation.

RANDOM DAMAGE GROWTH UNDER UNIAXIAL LOADING

Under uniaxial loading, (15) reduces to

$$\sigma_\infty + \psi_D \frac{dD}{d\varepsilon} + s_b = 0 \tag{16}$$

where σ_∞ is the far-field stress (possibly a random process) acting normal to the surface. The quantity s_b, which has dimensions of energy per unit volume per unit strain (or units of stress), may be interpreted as a random fluctuation imposed on the deterministic stress field existing within the deformable body. It is assumed that (1) s_b is a zero-mean process that assumes positive and negative values with equal probability; (2) the mean-square fluctuation is independent of strain (or time); and (3) the rate of fluctuation in s_b can be described as extremely rapid in comparison with the macroscopic rate of change in damage. The preceding assumptions are satisfied if s_b is described by the Langevin equation

$$\frac{ds_b}{d\varepsilon} = -c_1 s_b + \sqrt{c_2}\xi \tag{17}$$

where $\xi(\varepsilon)$ = Gaussian white noise (indexed with strain); and c_1, c_2 = positive constants. For a sufficiently large value of c_1, the process s_b becomes stationary with variance $c_2/(2c_1)$. Since the scale of fluctuations in s_b are short [(17)] compared to the scale of the index parameter (time or strain) of interest in structural mechanics [(16)], the following SDE for damage growth (Gardiner 1985) can be written as:

$$dD(\varepsilon) = -\frac{\sigma_\infty}{\psi_D} d\varepsilon - \frac{\sqrt{c_2}/c_1}{\psi_D} dW(\varepsilon) \tag{18}$$

where $W(\varepsilon)$ = standard Wiener process. Proceeding similarly, the random damage growth may be indexed with time, rather than with strain, if the strain rate $\dot{\varepsilon}$ is known

$$dD(t) = -\frac{\sigma_\infty}{\psi_D} \dot{\varepsilon}(D, t) \, dt - \frac{\sqrt{c_4}/c_3}{\psi_D} \dot{\varepsilon}(D, t) \, dW(t) \tag{19}$$

where c_3, c_4 = positive constants defining a Langevin equation similar to (17). The initial damage D_0 to be used as the initial condition in (18) or (19) is, in general, a random variable that takes into account the effects of residual stresses, surface roughness, loading histories, etc.

For uniaxial monotonic loading, the free energy per unit volume is

$$\psi = \int \sigma \, d\varepsilon - \gamma \tag{20}$$

where γ denotes the energy of formation of discontinuities per unit volume due to damage growth, estimated as (Bhattacharya 1997)

$$\gamma = \frac{3}{4}\sigma_f D \tag{21}$$

under the assumptions that (1) the discontinuities are microscopic spheres of different sizes that do not interact with each other; (2) the force-displacement relation is linear at the microscale; and (3) stress amplification effects can be neglected. The term σ_f is the true failure stress.

In the following sections, available experimental data are used to validate the theoretical stochastic damage accumulation models described previously. The constitutive model for the analysis of ductile deformation is defined by the Ramberg-Osgood law, $\varepsilon = \tilde{\sigma}/E + (\tilde{\sigma}/K)^M$, to describe the relation between effective stress and total strain under uniaxial monotonic loading. This law decomposes the total strain ε into its elastic (ε_e) and plastic (ε_p) components, with parameters E (the elastic modulus) and K and M (the hardening modulus and exponent, respectively). It is assumed that the exponent M is unaffected by damage. The constitutive model for fatigue damage is similarly defined by the cyclic Ramberg-Osgood law with parameters E, K', and M', which must be obtained from a stabilized cyclic stress-strain curve (Dowling 1993). The Bailey-Norton law provides the constitutive model for the analysis of creep damage, as described subsequently (Dowling 1993). Consistent with the experimental conditions, the applied stresses are assumed to be deterministic. However, the model is capable of handling randomness in the load by treating σ_∞ in (18) [or (19)] as a random process.

The preceding formulation of damage growth does admit negative damage increments, the probability of which depends on the relative magnitude of the drift and diffusion terms. Even though local and transient retardation in damage might actually occur at the microscale, the increment of damage should be nonnegative over a finite interval of time and space, in the absence of repair or autogenous healing. This property should be verified in every situation where the model is applied.

Stochastic Ductile Damage Growth

The free energy per unit volume can be written as

$$\psi = \int \tilde{E}\varepsilon_e \, d\varepsilon_e + \int \tilde{K}\varepsilon_p^{1/M} \, d\varepsilon_p - \frac{3}{4}\sigma_f D \tag{22}$$

where the damaged moduli, $\tilde{E} = E(1 - D)$ and $\tilde{K} = K(1 - D)$ for $\varepsilon_p \geq \varepsilon_0$, and ε_0 = threshold plastic strain for damage initiation (Lemaitre 1985). Eq. (18) can be written as

$$dD(\varepsilon_p) = \frac{\varepsilon_p^{1/M}(1 - D(\varepsilon_p))}{\frac{\varepsilon_p^{1+1/M}}{1 + 1/M} + C} d\varepsilon_p + \frac{(\sqrt{c_2}/c_1)/K}{\frac{\varepsilon^{1+1/M}}{1 + 1/M} + C} dW(\varepsilon_p) \quad (23)$$

where two simplifications have been incorporated: (1) $d\varepsilon/d\varepsilon_p \simeq 1$, which is true for all ε of interest in ductile deformation damage; and (2) $K/(2E) \sim 0$, which is valid for most engineering alloys. The constant in (23) is

$$C = \frac{3}{4}\frac{\sigma_f}{K} - \varepsilon_0^{1+1/M} \Big/ \left(1 + \frac{1}{M}\right) \quad (24)$$

and is in general a random variable on account of material variability.

TABLE 1. Material Properties for 2024-T3 Aluminum

Parameter (1)	Nominal (2)	Mean (3)	COV (4)	Distribution[c] (5)
E	74.5 GPa[a]	—	—	—
K	680 MPa[a]	680 MPa	0.20[c]	Normal
M	5.5[a]	5.5	0.20[c]	Normal
σ_f	435 MPa[a]	435 MPa	0.20[c]	Normal
ε_0	0.016[b]	0.016	1.0[d]	Lognormal
D_0	0[c]	—	—	Deterministic
$\sqrt{c_2}/c_1$	20 MPa[c]	—	—	Deterministic
D_c	0.23[b]	0.23	0.10[c]	Normal

[a]See Hansen and Schreyer (1994).
[b]See Lemaitre (1985).
[c]Assumed.
[d]See Woo and Li (1993).

Eq. (23) is of the form of a time-dependent Ornstein-Uhlenbeck process, and because the diffusion term is independent of D, its Ito and Stratonovich solutions are identical (Gardiner 1985)

$$D(\varepsilon_p) = 1 - (1 - D_0)\frac{(3/4)(\sigma_f/K)(1 + 1/M)}{\varepsilon_p^{1+1/M} + (1 + 1/M)C} + \frac{(\sqrt{c_2}/c_1)/K}{\frac{\varepsilon^{1+1/M}}{1 + 1/M} + C}[W(\varepsilon_p) - W(\varepsilon_0)] \quad (25)$$

where $D_0 = D(\varepsilon_0)$ = the initial damage.

Numerical Examples

To validate the proposed CDM model, the experimental results from Woo and Li (1993) on the mean and standard deviation of random ductile damage growth in 2024-T3 Aluminum were utilized. Damage growth data from Lemaitre (1985) on the same nominal grade of material are also included in the validation; this data set, however, does not contain any statistical description. The nominal values and statistical properties of the material parameters are listed in Table 1. The initial damage is assumed to equal zero to conform to the way the experiments were carried out. The applied stress was increased monotonically from zero. The quantity $\sqrt{c_2}/c_1$, which is related to the ratio of the variance and the correlation length of the fluctuating quality s_b, was selected to model the overall magnitude of the observed standard deviation of damage (Woo and Li 1993). The predicted mean and standard deviation functions of damage are obtained numerically from (25).

In Fig. 1(a), the material properties, $\underline{\Omega} = \{\ln(\varepsilon_0), \sigma_f, K, M\}$,

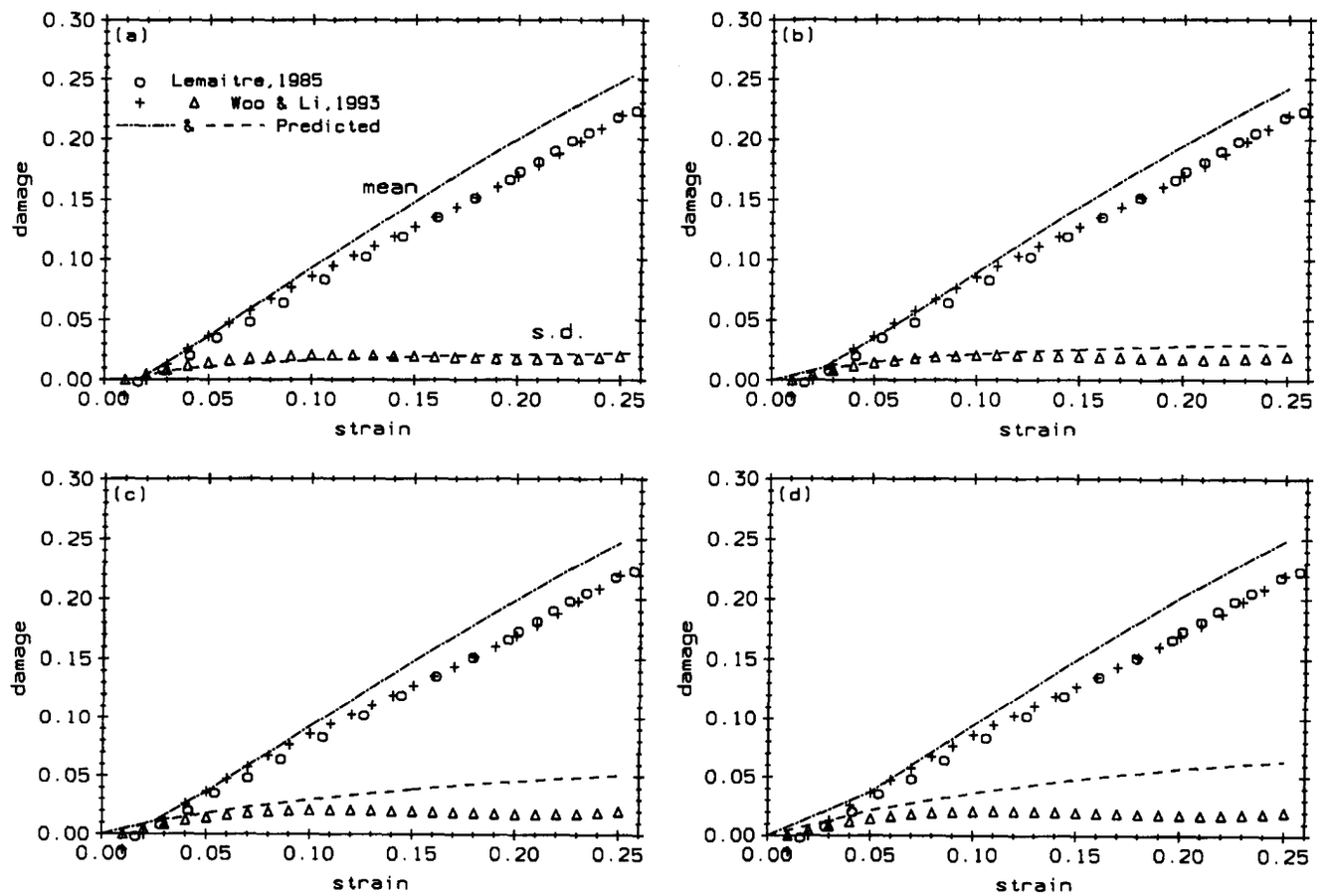

FIG. 1. Effect of Randomness and Correlation among Variables: (a) All Deterministic; (b) Perfect Correlation; (c) Moderate Correlation; (d) Zero Correlation

are assumed to be nonrandom (equal to their nominal or handbook values) and the randomness in damage growth is due solely to noise in the process. The close agreement of the predicted functions with the experiment confirms the validity of the assumption underlying (7). In Figs. 1(b–d) the material properties, $\underline{\Omega} = \{\ln(\varepsilon_0), \sigma_f, K, M\}$, are considered random with varying degrees of stochastic dependence among them. The means of the random variables are assumed equal to their nominal values. When the random variables are perfectly correlated [Fig. 1(b)], the correlation matrix of $\{\ln(\varepsilon_0), \sigma_f, K, M\}$ is the identity matrix; and when moderately correlated [Fig. 1(c)], the off-diagonal terms of the correlation matrix are all taken to be 0.5. Treating the parameters as random and varying the correlation among them has almost no effect on the mean damage accumulated. However, the standard deviation of $D(\varepsilon_p)$ is significantly affected by randomness in all variables and is the highest when the random variables are considered to be mutually statistically independent [Fig. 1(d)]. While considering these random variables to be statistically independent would be a simplifying and conservative assumption, it may overestimate the scatter in damage by a factor of 2.

Excellent agreement with experimental results can be achieved with several combinations of the noise intensity (Bhattacharya 1997), correlation coefficients, and the marginal distributions suggesting that, with proper parameter identification, this method can reproduce experimentally observed stochastic damage accumulation.

Reliability in Ductile Deformation

Failure occurs when damage reaches the critical value D_c, which corresponds to localization in the damage growth process. The cumulative failure probability (CFP) at strain ε, $F_{\varepsilon_f}(\varepsilon)$, is

$$F_{\varepsilon_f}(\varepsilon) = 1 - P[D(\varepsilon') \leq D_c;\ \forall \varepsilon' \in [0, \varepsilon]] \tag{26}$$

where ε_f = random failure strain. Mathematically, this is a first passage problem.

If damage growth rate is almost always positive, the sample paths of $D(\varepsilon)$ which cross D_c (from below) for the first time at $\varepsilon_1 < \varepsilon$, may be expected to stay above that barrier at ε. This becomes more likely as the interval $(\varepsilon - \varepsilon_1)$ becomes larger. In such cases, the CFP [(26)] can be simplified as the complement of the CDF of the damage function evaluated at the critical damage

$$F_{\varepsilon_f}(\varepsilon) \simeq 1 - P[D(\varepsilon) \leq D_c] \tag{27}$$

For random ductile damage accumulation in 2024-T3 Aluminum, the validity of (27) is shown in Bhattacharya (1997). Fixing the initial condition and the material parameters, $\underline{\Omega}_1 = \{D_0, \ln(\varepsilon_0), \sigma_f, K, M, D_c\}$, the CFP under assumption (27), can be written as

$$F_{\varepsilon_f|\underline{\Omega}_1}(\varepsilon) = 1 - \Phi\left(\frac{D_c - \mu_{D|\underline{\Omega}_1}(\varepsilon)}{\sigma_{D|\underline{\Omega}_1}(\varepsilon)}\right) \tag{28}$$

where $\mu_{D|\underline{\Omega}_1}(\varepsilon)$ and $\sigma_{D|\underline{\Omega}_1}(\varepsilon)$ = the conditional mean and standard deviation, respectively. The theorem of total probability may be used to remove the conditioning on $\underline{\Omega}_1$ if their joint probability density is known.

Fig. 2 illustrates the limit state probability (26) for 2024-T3 Aluminum, in which D_c is treated as a random variable (with parameters defined in Table 1). Parameters $\{\ln(\varepsilon_0), \sigma_f, K, M\}$ are considered random as before (Table 1), with correlation coefficient of 0.5 between each pair. The noise intensity is 20 MPa. To give a visual sense of the scatter in the damage accumulation process, a few sample functions of $D(\varepsilon)$ (selected at random) are shown in Fig. 2; these sample functions were obtained numerically from (23) using an interval size $\Delta\varepsilon$ =

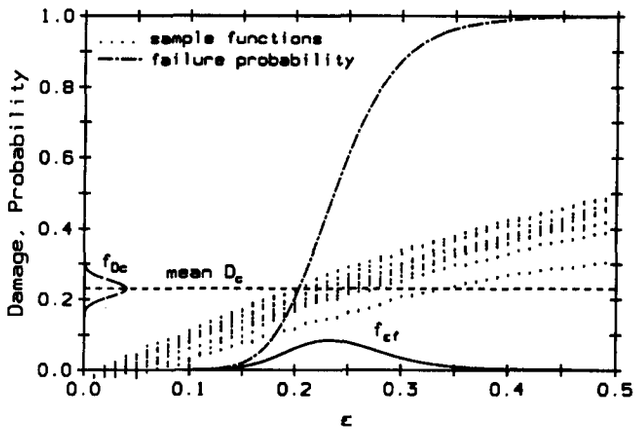

FIG. 2. Failure Probability and Sample Paths of Ductile Damage in 2024-T3 Aluminum

0.01. No sample function returns to the safe region once it has exited that region, reinforcing the notion of nonnegative damage growth. The relation between D and ε cannot be determined explicitly [(27)], and the CDF of ε_f must be obtained numerically, following which its probability density function is obtained by numerical differentiation. The mean and standard deviation of ε_f are found to be, respectively, 0.247 and 0.052—values which are of the same order as those generally observed for engineering metals [e.g., Davis (1993)].

Stochastic Fatigue Damage

Random Fatigue Damage Accumulation

Fatigue damage growth is modeled as occurring incrementally in successive load cycles until crack initiation occurs

$$D_{i+1} = D_i + \Delta D_i;\quad \Delta D_i \geq 0,\quad i = 1, \ldots, N_I - 1 \tag{29}$$

$$D_{N_I - 1} < D_c;\quad D_{N_I} \geq D_c \tag{30}$$

Fatigue damage in a given cycle is assumed to accumulate only during the loading portion of the stress-strain loop when the stress is greater than the endurance limit S_e. Damage growth in cycle i may therefore be described by the random monotonic damage growth equation (23) with the damage at the end of the previous cycle taken as the initial damage

$$\frac{dD}{d\varepsilon} = \begin{cases} -\dfrac{\sigma_\infty}{\psi D} - \dfrac{\sqrt{c_2}}{c_1}\dfrac{\xi(\varepsilon)}{\psi_D}; & \sigma_\infty \geq S_e \geq 0,\quad \dot{\varepsilon} > 0 \\ 0; & \text{otherwise} \end{cases} \tag{31}$$

Note that (31) can predict the random characteristics of only the initiation phase (prior to macrocrack formation), and any inferences drawn therefrom should not be extrapolated to the crack propagation phase.

The SDE describing fatigue damage growth in cycle i and indexed by the plastic strain range $\Delta\varepsilon_p$ is:

$$dD = a(1 - D)\, d\Delta\varepsilon_p + b\, dW(\Delta\varepsilon_p) \tag{32}$$

with the initial condition, $D = D_{i-1}$ at $\Delta\varepsilon_p = \Delta\varepsilon_{p0_i}$, where $\Delta\varepsilon_{p0_i}$ is the plastic strain range corresponding to $\sigma = S_e$. The coefficients are

$$a(\Delta\varepsilon_p) = \frac{\Delta\varepsilon_p^{1/M'} - \Delta\varepsilon_{p1_i}^{1/M'}}{\dfrac{\Delta\varepsilon_p^{1+1/M'}}{1 + 1/M'} - \Delta\varepsilon_p\Delta\varepsilon_{p1_i}^{1/M'} + C_i} \tag{33}$$

$$b(\Delta\varepsilon_p) = \frac{(\sqrt{c_2}/c_1)/K_1}{\dfrac{\Delta\varepsilon_p^{1+1/M'}}{1 + 1/M'} - \Delta\varepsilon_p\Delta\varepsilon_{p1_i}^{1/M'} + C_i} \tag{34}$$

$$C_i = \frac{3}{4}\frac{\sigma_f}{K_1} - \frac{\Delta\varepsilon_{p0_i}^{1+1/M'}}{1 + 1/M'} + \Delta\varepsilon_{p1_i}^{1/M'}\Delta\varepsilon_{p0_i} \tag{35}$$

where $\Delta\varepsilon_{p1_i}$ = plastic strain range corresponding to $\varepsilon = 0$, $K_1 = 2^{1-1/M'}K'$, and it has been assumed that $K_1/E \sim 0$ as before. It may be recalled that E, K', and M' are the Ramberg-Osgood cyclic stress-strain parameters. The surface energy increment in cycle i, given by $(3/4)\sigma_f(D - D_{i-1})$. The damage, D_i, at the end of cycle i is the solution of (32) at $\Delta\varepsilon_p = \Delta\varepsilon_{pm_i}$, where $\Delta\varepsilon_{pm_i}$ is the maximum plastic strain range in cycle i. The drift term b in (32) is independent of damage D. Consequently, the Ito and Stratonovich solutions (Gardiner 1985) of (32) are identical

$$D_i = 1 - \frac{(1 - D_{i-1})(3/4)(\sigma_f/K_1)}{\dfrac{\Delta\varepsilon_{pm_i}^{1+1/M'}}{1 + 1/M'} - \Delta\varepsilon_{pm_i}\Delta\varepsilon_{p1_i}^{1/M'} + C_i} + \frac{(\sqrt{c_2}/c_1)\Delta W_i/K_1}{\dfrac{\Delta\varepsilon_{pm_i}^{1+1/M'}}{1 + 1/M'} - \Delta\varepsilon_{pm_i}\Delta\varepsilon_{p1_i}^{1/M'} + C_i} \tag{36}$$

where ΔW_i = increment of the standard Wiener process over the cycle i. The recursive nature of $(1 - D_i)$ in (36) makes it possible to express damage D_n at the end of n cycles in terms of the initial damage D_0 and n independent increments of the standard Wiener process

$$D_n = 1 - (1 - D_0)\left(\frac{3\sigma_f}{4K_1}\right)^n \prod_{i=1}^{n} g_i + C_0 \sum_{i=1}^{n} \Delta W_i \prod_{j=i}^{n}\left(\frac{3\sigma_f}{4K_1}\right) g_j \tag{37}$$

where

$$g_i = \frac{1}{\dfrac{\Delta\varepsilon_{pm_i}^{1+1/M'}}{1 + 1/M'} - \Delta\varepsilon_{pm_i}\Delta\varepsilon_{p1_i}^{1/M'} + C_i} \tag{38}$$

In some situations it may be more convenient to express damage as a function of time rather than number of cycles. In this case the functional form of $n(t)$ (including its stochastic characteristics) must be incorporated in (37).

Comparisons with Fatigue Crack Initiation Experiments

Assuming that the damage growth process described by (37) is almost always positive, the cumulative probability of failure is the complement of the CDF of D_n

$$P[N_I \le n] = F_{N_I}(n) \simeq P[D_n > D_c] \tag{39}$$

from which the probability distribution of N_I may be obtained if the statistics of D_n and D_c are known.

No published data on random fatigue damage growth (during the preinitiation stage) oriented toward CDM could be located. Limited data are available on random initiation life [e.g., Majumdar et al. (1993)], which allow partial validation of the present model. In the following, random fatigue damage growth in A106-B (a carbon steel) subjected to fully reversed cycling at 288°C in air is considered. The material properties, E = 196.5 GPa, K' = 1,994 MPa, M' = 7.74, σ_f = 539 MPa, σ_y = 301 MPa, and S_e = 310 MPa (taken from Chopra et al. 1995) are treated deterministically. The noise parameter, $\sqrt{c_2}/c_1$ = 1,000 MPa, is derived (as described in Bhattacharya 1997) to match the standard deviation observed in fatigue tests conducted by Keisler et al. (1994).

The fatigue crack initiation time N_I and propagation time N_P are both random, and the statistics of their sum, $N_T = N_I + N_P$, depends on their correlation structure. The variability in N_I is due to the fluctuations in energy and material properties throughout the material volume (a necessary assumption of CDM), whereas the variability in N_P depends solely on the local fluctuations around the crack-tip (which is the basic tenet of fracture mechanics). Therefore, the probabilistic estimation of N_T can be simplified through the assumption that N_I and N_P are independent of one another.

Fig. 3 shows the predicted mean time to crack initiation μ_{N_I} [obtained using (39) with D_c = 0.25 and D_0 = 0] along with an estimate of scatter in the initiation time. The predicted range compares favorably with the deterministic estimate of N_I from Majumdar et al. (1993) that corresponds to the formation of a 0.18-mm-long crack. Fig. 3 also provides an estimate of the total fatigue life N_T as the sum, $\mu_{N_I} + N_P$, with the (deterministic) propagation life obtained by integrating the Paris's law between the limits $a_s = (1/\pi)(\Delta K_{th}/2S_e)^2$ and a_f = 6.35 mm, subject to the condition $K_{max} \le \min(K_c, \sqrt{(E\sigma_y\delta_T)})$. The material properties used are: K_c = 66 MPa$\sqrt{\text{m}}$, ΔK_{th} = 6.0 MPa$\sqrt{\text{m}}$, δ_T = 0.04 mm; the Paris's law parameters are $C = 6.9 \times 10^{-9}$ mm/cycle and m = 3 (Barsom and Rolfe 1987; Dowling 1993). The prediction of total fatigue life shown earlier agrees well with the observed cycles to failure N_T (O. K. Chopra, private communication, 1996), where failure was defined as a 25% drop in the peak stress; this corresponds to the formation of a 3-mm crack and is quite close to the point of rupture. The proposed model also is consistent with the Coffin-Manson model of strain-controlled fatigue, which separates the total strain-total life (ε_a-N) relation into its elastic (ε_{ea}-N) and plastic (ε_{pa}-N) components.

Fig. 4 illustrates the predicted number of cycles for three different possibilities of crack initiation over a range of strain amplitudes for the same material and test conditions and compared them with the experimental number of cycles to failure (O. K. Chopra, private communication, 1996). The predicted probabilities of crack initiation agree qualitatively with those reported by Keisler et al. (1994), where a family of P-ε_a-N curves corresponding to the formation of a 3-mm crack in a smooth specimen are plotted. The 5% curve from Keisler et al. (1994), which includes the crack growth phase, is reproduced in Fig. 4 for comparison.

The randomness in fatigue life is a function of the initiation and propagation stages, and a consideration of both aspects is needed for its complete characterization. Available models of random fatigue crack growth [e.g., Virkler et al. (1979)] derived from experiments with precracked specimens ignore the fundamental problem of when the crack may initiate in a smooth specimen. Figs. 3 and 4 illustrate that stochastic CDM shows promise for reducing the empiricism of current approaches to predicting crack initiation.

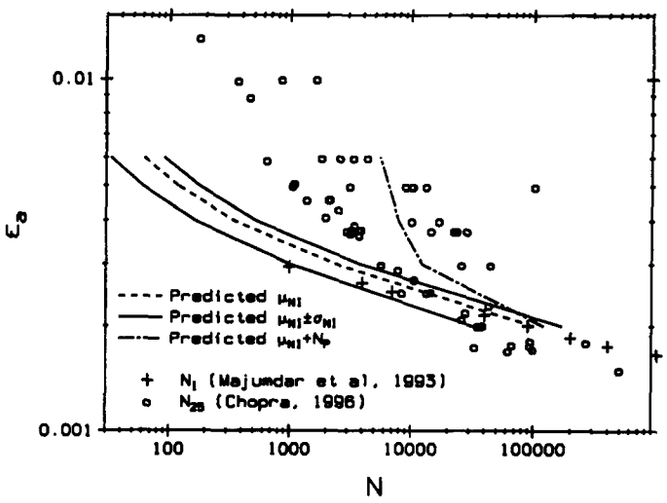

FIG. 3. Random Number of Cycles to Failure (Fatigue Crack Initiation) in A106-B Carbon Steel at 288°C in Air (O. K. Chopra, Private Communication, 1996)

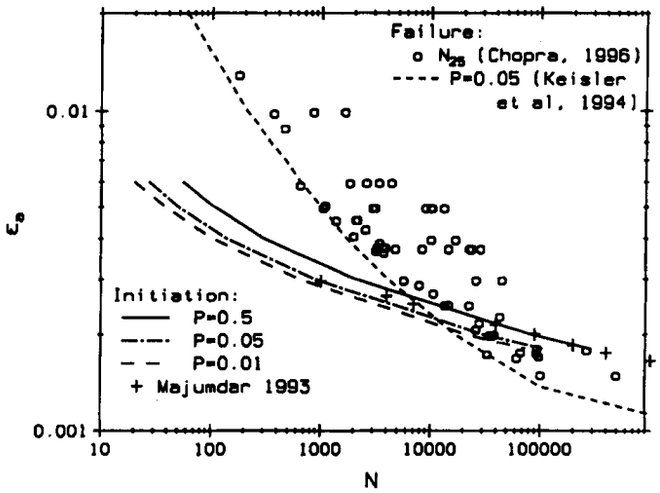

FIG. 4. Probability of failure (Fatigue Crack Initiation) in A106-B Carbon Steel at 288°C in Air (O. K. Chopra, Private Communication, 1996)

Stochastic Creep Damage

Prediction of Damage Growth

The equivalence principle [cf. (1) and (2)] applied to the Bailey-Norton creep strain rate (Dowling 1993) under uniaxial loading yields

$$\dot{\varepsilon}_{cr} = A\phi\tilde{\sigma}^m t^{\phi-1} \tag{40}$$

where $\dot{\varepsilon}_{cr}$ = creep strain rate; and $\tilde{\sigma}$ = effective applied stress. Under constant stress creep, the increment in total strain is the same as that in creep strain, i.e., $\dot{\varepsilon} = \dot{\varepsilon}_{cr}$, and ψ_D can be simplified as [cf. (20) and (21)]

$$\psi_D = -\frac{3}{4}\sigma_f \tag{41}$$

where σ_f = true failure stress at the operating temperature. The stochastic differential equation of isotropic creep damage growth can be written as

$$dD(t) = \frac{A_1}{(1 - D(t))^m}\,dt + \frac{B_1}{(1 - D(t))^m}\,dW(t) \tag{42}$$

where the coefficients A_1 and B_1 are

$$A_1 = \frac{4}{3}\frac{A\phi t^{\phi-1}}{\sigma_f}\sigma_\infty^{m+1} \tag{43}$$

$$B_1 = \frac{4}{3}\frac{A\phi t^{\phi-1}}{\sigma_f}\sigma_\infty^m(\sqrt{c_4}/c_3) \tag{44}$$

To the knowledge of the writers, (42) does not have a closed-form solution in the Ito sense. But a closed-form solution is possible in the Stratonovich sense, under the condition $\phi = 1$ (steady-state creep)

$$(1 - D(t))^{m+1} = (1 - D_0)^{m+1} - A_1 t(m + 1) - B_1(m + 1)W(t) \tag{45}$$

where the initial time $t_0 = 0$. The initial damage D_0 takes into account the ductile damage caused when the component is loaded to σ_∞ at the beginning of the process, in addition to any damage existing prior to the commencement of creep straining. Eq. (45) can be rearranged as

$$D(t) = 1 - \{(1 - D_0)^{m+1} - A_1(m + 1)t\}^{1/(1+m)} \cdot \left[1 - \frac{B_1(m + 1)W(t)}{(1 - D_0)^{m+1} - A_1 t(m + 1)}\right]^{1/(1+m)} \tag{46}$$

which has the same form as the deterministic solution for steady-state creep (Bhattacharya 1997), with an additional term (in brackets) containing noise.

Sample functions of $D(t)$ are absorbed by the boundary $D = 1$ as t approaches the failure time. The cumulative distribution function (CDF) of $D(t)$, is therefore a mixed distribution in [0, 1], with the probability $P[D(t) = 1]$ increasing with time. In the open interval (0,1), the CDF of $D(t)$, conditioned on fixed values of the initial condition and material properties, $\underline{\Omega} = \{D_0, A, \sigma_f, m\}$, can be written as

$$F_{D|\underline{\Omega}}(d; t) = \Phi\left[\frac{p(t) - (1 - d)^{m+1}}{q(t)}\right], \quad 0 < d < 1 \tag{47}$$

The finite probabilities at 0 and 1 are

$$P_0(t) = P[D(t) = 0] = \Phi\left[\frac{p(t) - 1}{q(t)}\right] \tag{48}$$

$$P_1(t) = P[D(t) = 1] = 1 - \Phi\left[\frac{p(t)}{q(t)}\right] \tag{49}$$

where $\Phi(\cdot)$ = standard normal distribution function and

$$p(t) = (1 - d_0)^{m+1} - A_1 t(m + 1) \tag{50}$$

$$q(t) = B_1(m + 1)\sqrt{t} \tag{51}$$

Here, d_0 = fixed value of the random variable D_0. If the initial condition and material parameters are random and their joint probability density is known, the unconditional moments and the unconditional CDF can be obtained using the theorem of total probability.

Model Verification and Studies in Reliability

The cumulative failure probability in the interval $[0, t]$ is

$$F_T(t) = 1 - P[D(\tau) < D_c \forall \tau \in [0, t]] \tag{52}$$

where T = random time to failure. Arguing as in the case of stochastic ductile damage

$$F_T(t) \simeq P_f(t) = P[D(t) > D_c] \tag{53}$$

provided that the damage growth rate is almost always positive.

No published CDM-based studies of stochastic creep damage growth (e.g., by measuring the reduced stiffness) could be located. Therefore, in the following numerical examples on creep damage growth in Type 316 stainless steel stressed to 199 MPa at 593°C (1,100°F), only the statistics of the predicted failure time are compared with available experimental results. The normal creep law and tensile parameters for Type 316 stainless steel at 593°C are listed in Table 2 (Garofalo et al. 1961; Davis 1994). When a material parameter is treated as random in the following, its mean is taken equal to its nominal value. Parameters are considered statistically independent of each other, when treated as random. The nominal value of D_0 is computed from (25) with deterministic E, K, σ_f, M, zero prior damage, and zero noise.

Figs. 5(a–e) show the effect on the predicted mean and the (predicted) standard deviation of damage and the failure probability (53) under the Stratonovich interpretation of (42). Parameters treated as random are indicated in each figure, along with the mean μ_T and coefficient of variation (COV), V_T of the failure time. A 10% variability in the initial damage causes no significant increase in the failure probability as a function of time [cf. Figs. 5(a) and 5(b)]. A 20% variability in the rate, A, causes V_T to rise from about 5.3 to 20.3%. Introducing a 10% variability in D_c has almost no effect on the failure probability at this stage. However, introducing a mere 1% variability to m causes V_T to jump to 43.3%. These values may be

TABLE 2. Creep and Tensile Properties of Type 316 Stainless Steel at 593°C

Parameters (1)	A (MPa/h) (2)	m (3)	ϕ (4)	Test σ (MPa) (5)	σ_f (MPa) (6)	E (GPa) (7)	K (MPa) (8)	M (9)	D_0 (10)	D_c (11)	$\sqrt{c_4/c_3}$ (MPa$\sqrt{h}$) (12)
Nominal	2.32×10^{-20}[b]	6.92[b]	1[b]	199–315[b]	443.7[c]	151.6[c]	492.7[d]	4.22[d]	0.0108	0.20[a]	300[a]
COV[a]	0.20	0.01	—	—	—	—	—	—	0.10	0.10	—
Distribution[a]	LN	N	det	det	det	det	det	det	N	N	det

Note: LN = lognormal, N = normal, det = deterministic.
[a]Assumed.
[b]See Garofalo et al. (1961).
[c]See Davis (1994).
[d]See Boyer (1987).

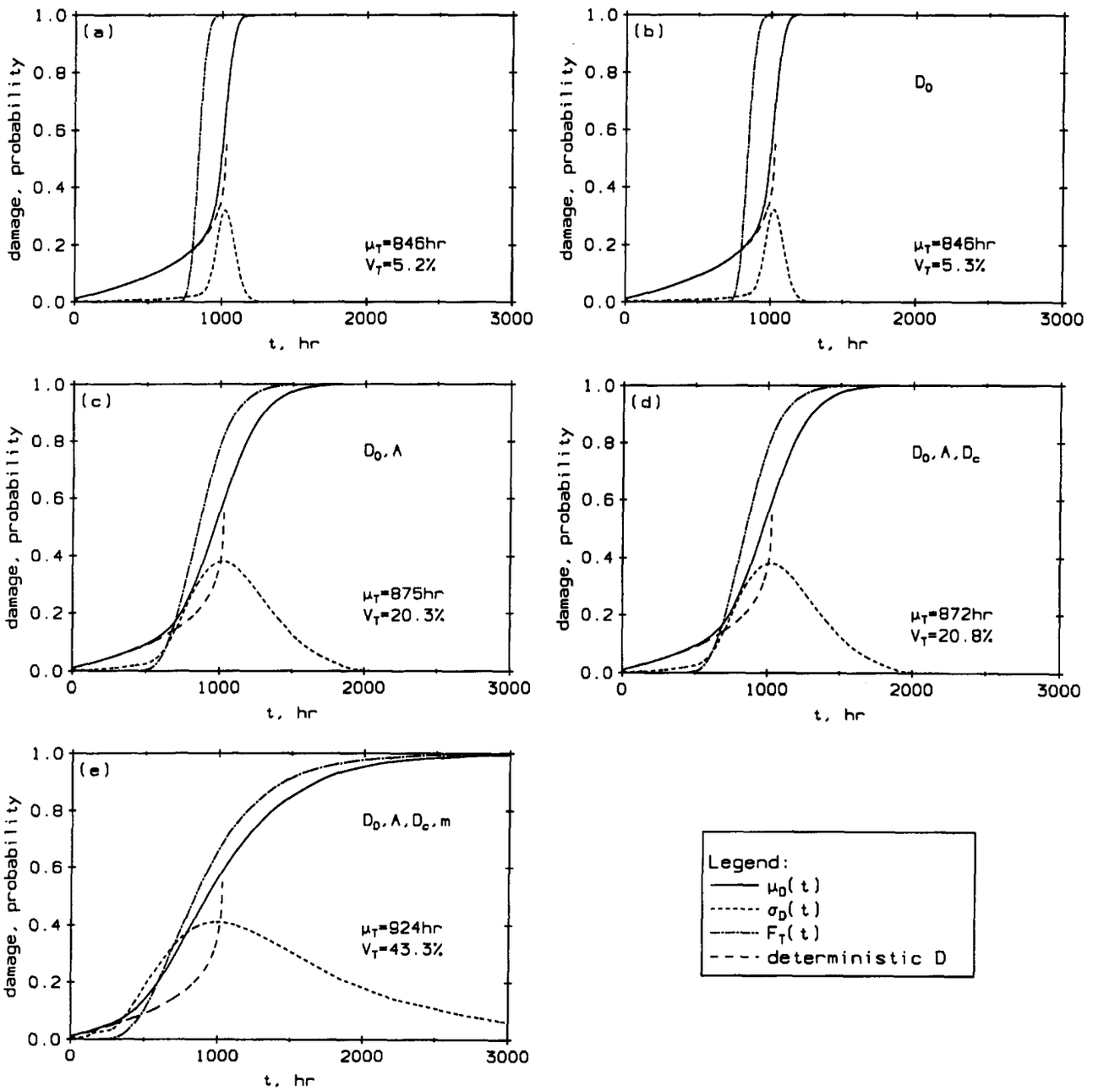

FIG. 5. Stratonovich Solutions of Random Creep Damage in Type 316 Stainless Steel: (a) All Deterministic; (b) D_0 Random; (c) D_0, A Random; (d) D_0, A, D_c Random; (e) All Random

ability to m causes V_T to jump to 43.3%. These values may be compared with the scatter observed by Garofalo et al. (1961) in the times (hours) to (1) the onset of tertiary creep (mean = 1,283, COV = 0.28, min = 960, max = 1,950); and (2) rupture (mean = 1,749, COV = 0.21, min = 1,267, max = 2,437); under the same conditions of temperature and stress.

Figs. 6(a–e) show the mean, standard deviation, and cumulative failure probability (52) of creep damage in the same

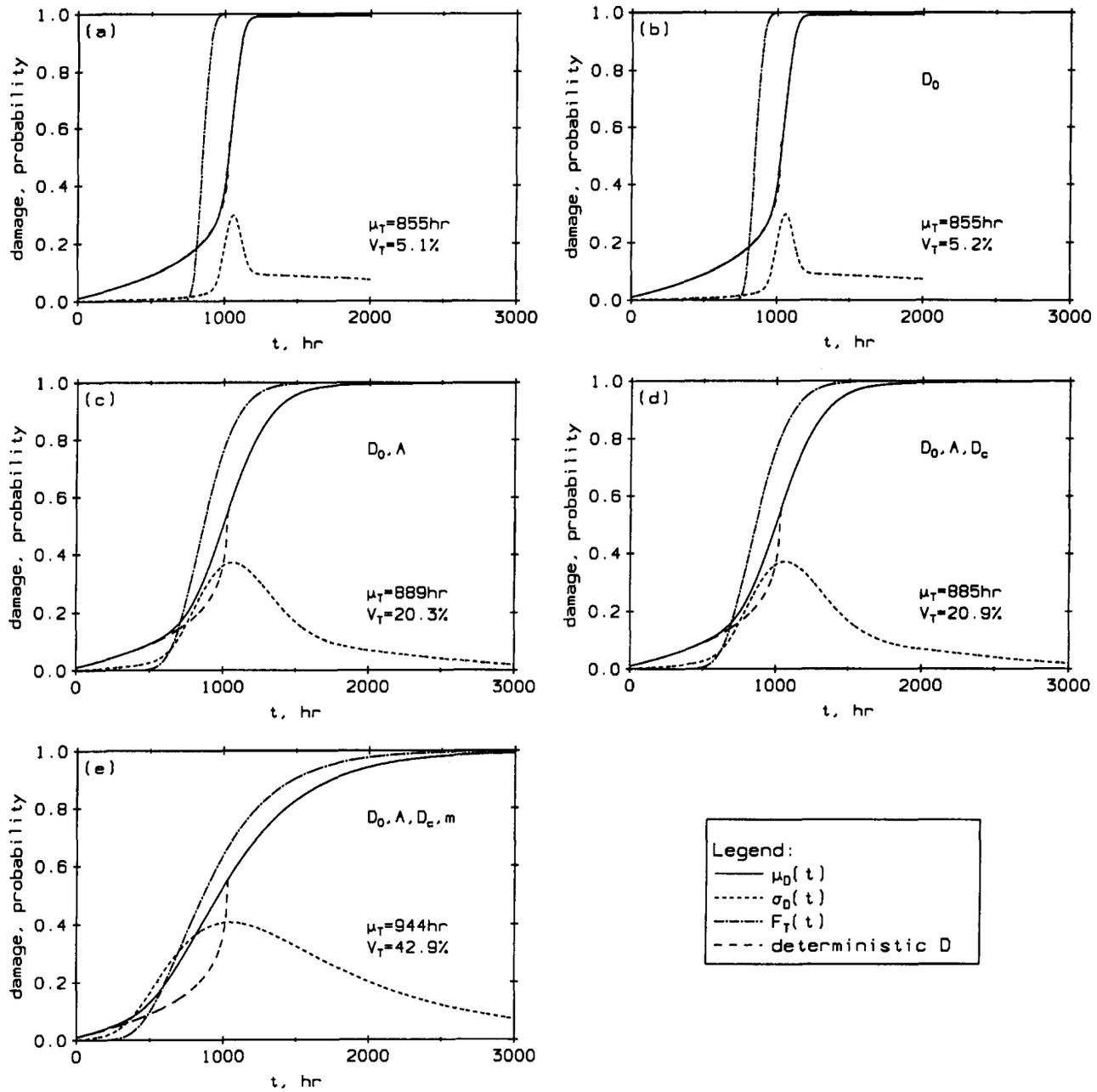

FIG. 6. Ito Solutions of Random Creep Damage in Type 316 Stainless Steel: (a) All Deterministic; (b) D_0 Random; (c) D_0, A Random; (d) D_0, A, D_c Random; (e) All Random

material and loading condition as shown earlier, using the Ito interpretation of (42). Functions $\mu_D(t)$, $\sigma_D(t)$, and $F_T(t)$ are computed numerically using a uniform time step of 10 h. A comparison of the two approaches (Ito and Stratonovich) in the five different scenarios shows that the latter results in a marginal decrease in μ_T but provides a correspondingly higher estimate of the COV V_T. The Stratonovich solution, which is computationally more efficient, therefore, provides a slightly more conservative estimate of reliability than the Ito solution.

In both Figs. 5 and 6, the standard deviation of damage is found to grow with time until the mean damage is close to 1; it subsequently starts to decrease, which is due to the fact that $D = 1$ is an absorbing boundary. It should be noted that the failure probabilities in Fig. 5 are obtained from (53); those in Fig. 6 are obtained from the numerical solution of (52), which is computationally more involved. However, the close agreement of the two different approaches confirms the accuracy of the approximate equation (53). Moreover, the deterministic solutions for the same condition of stress and temperature [obtained using the nominal values of the material parameters, and $c_4 = 0$ for zero noise in (46)] agree closely with the predicted mean damages.

CONCLUSIONS

A set of SDEs of damage growth prior to localization in a deformable body is obtained from first principles of thermodynamics. The proposed approach deals with several aspects of randomness: the initial damage conditions, intrinsic microstructural variations, and variabilities in the macroscopic material properties, all of which affect the current state as well as the growth rate of the damage process. A detailed knowledge of all sources of randomness is required for a satisfactory description of stochastic damage growth and reliability. The CDM-based model developed here provides a unified approach to ductile deformation, fatigue, and creep damage processes.

Agreement with available experimental results by other researchers was very good.

Additional experimental data to further validate the model would be desirable to enhance its use as a tool for condition assessment and service life prediction of structural components and systems. Further research should also be directed at including loading-rate effects on the damage growth process. Efforts are underway to provide a priori estimates of the Langevin equation constants [(17)] from mechanistic/thermodynamic considerations.

ACKNOWLEDGMENTS

Support for the research, provided, in part, by Lockheed-Martin Energy Research Corporation under Grant 19X-SP638V, with Dr. D. J. Naus as Program Manager, is gratefully acknowledged. The writers would also like to thank Omesh Chopra of Argonne National Laboratory, Argonne, Ill., for making available the fatigue loading data used in this paper.

APPENDIX. REFERENCES

Barsom, J. M., and Rolfe, S. T. (1987). *Fracture and fatigue control in structures*. Prentice-Hall, Inc., Englewood Cliffs, N.J.

Bhattacharya, B. (1997). "A damage mechanics based approach to structural deterioration and reliability," PhD thesis, Johns Hopkins Univ., Baltimore, Md.

Bhattacharya, B., and Ellingwood, B. R. (1996). "A CDM-based approach to stochastic damage growth." *Proc., ASCE Specialty Conf. on Probabilistic Mech.*, ASCE, New York, 772–775.

Boyer, H. E. (1987). *Atlas of stress-strain curves*. ASM International, Materials Park, Ohio.

Breysse, D. (1990). "Probabilistic formulation of damage-evolution law of cementitious composites." *J. Engrg. Mech.*, ASCE, 116(7), 1489–1510.

Callen, H. B. (1985). *Thermodynamics and an introduction to thermostatics*. John Wiley & Sons, Inc., New York, N.Y.

Carmeliet, J., and Hens, H. (1994). "Probabilistic nonlocal damage model for continua with random field properties." *J. Engrg. Mech.*, ASCE, 120(10), 2013–2027.

Chaboche, J. L. (1988). "Continuum damage mechanics—I and II." *J. Appl. Mech.*, 55, 59–72.

Chopra, O. K., et al. (1995). "Environmentally assisted cracking in light water reactors." *Rep. No. NUREG/CR-4667*, U.S. Nuclear Regulatory Commission, Washington, D.C.

Chow, C. L., and Wei, Y. (1991). "A model of continuum damage mechanics for fatigue failure." *Int. J. Fracture*, 50(4), 301–316.

Davis, J. R. (1993). *Aluminum and aluminum alloys*. ASM International, Materials Park, Ohio.

Davis, J. R. (1994). *Stainless steels*. ASM International, Materials Park, Ohio.

Diao, X. (1995). "A statistical equation of damage evolution." *Engrg. Fracture Mech.*, 52(1), 33–42.

Dowling, N. E. (1993). *Mechanical behavior of materials*. Prentice-Hall, Inc., Englewood Cliffs, N.J.

Gardiner, C. W. (1985). *Handbook of stochastic methods for physics, chemistry and the natural sciences*. Springer-Verlag New York, Inc., New York, N.Y.

Garofalo, R. W., et al. (1961). "Creep and creep-rupture relationships in an austenitic stainless steel." *Trans. Metallurgical Soc. of AIME*, 221, 310–319.

Hansen, N. R., and Schreyer, H. L. (1994). "A thermodynamically consistent framework for theories of elastoplasticity coupled with damage." *Int. J. Solids and Struct.*, 31(3), 359–389.

Harlow, D. G., and Delph, T. J. (1995). "A computational probabilistic model for creep-damaging solids." *Computers and Struct.*, 54(1), 161–166.

Kachanov, L. M. (1986). *Introduction to continuum damage mechanics*. Martinus Nijhoff Publishers, Dordrecht, The Netherlands.

Kandarpa, S., Kirkner, D. J., and Spencer, B. F., Jr. (1996). "Stochastic damage model for brittle materials subjected to monotonic loading." *J. Engrg. Mech.*, ASCE, 122(8), 788–795.

Keisler, J., Chopra, O. K., and Shack, W. J. (1994). "Statistical analysis of fatigue strain-life data for carbon and low-alloy steels." *Rep. No. NUREG/CR-6237, ANL-94/21*, Argonne Nat. Lab., Argonne, Ill.

Krajcinovic, D. (1984). "Continuum damage mechanics." *Appl. Mech. Rev.*, 37(1), 1–6.

Krajcinovic, D., and Mastilovic, S. (1995). "Some fundamental issues of damage mechanics." *Mech. of Mat.*, 21(3), 217–230.

Krajcinovic, D., and Silva, M. A. G. (1982). "Statistical aspects of the continuous damage theory." *Int. J. Solids and Struct.*, 18(7), 551–562.

Krajcinovic, D., and Sumarac, D. (1987). "Micromechanics of the damage process." *Continuum damage mechanics theory and applications*, D. Krajcinovic and J. Lemaitre, eds., Springer-Verlag New York, Inc., New York, N.Y.

Lemaitre, J. (1985). "A continuum damage mechanics model for ductile fracture." *J. Engrg. Mat. and Technol.*, 107(1), 83–89.

Lemaitre, J. (1992). *A course on damage mechanics*. Springer-Verlag New York, Inc., New York, N.Y.

Majumdar, S., Chopra, O. K., and Shack, W. J. (1993). "Interim failure design curves for carbon, low-alloy, and austenitic stainless steels in LWR environments." *Proc., 20th Water Reactor Safety Information Meeting, Report NUREG/OP-0126*, Vol. 3, U.S. Nuclear Regulatory Commission, Washington, D.C.

Ostoja-Starzewski, M. (1989). "Damage in a random microstructure: Size effects, fractals and entropy maximization." *Appl. Mech. Rev.*, 42(11), S202–S212.

Qian, Z., Takezono, S., and Tao, K. (1996). "A nonlocal damage mechanics approach to high temperature fatigue crack growth." *Engrg. Fracture Mech.*, 53(4), 535–543.

Sears, S. F., and Salinger, G. L. (1975). *Thermodynamics, kinetic theory and statistical thermodynamics*. Addison-Wesley Publishing Co., Reading, Mass.

Virkler, D. A., Hillberry, B. M., and Goel, P. K. (1979). "The statistical nature of fatigue crack propagation." *J. Engrg. Mat. and Technol.*, 101(2), 148–153.

Woo, C. W., and Li, D. L. (1992). "A general stochastic dynamic model of continuum damage mechanics." *Int. J. Solids and Struct.*, 29(23), 2921–2932.

Woo, C. W., and Li, D. L. (1993). "Statistical analysis of material damage with changing internal structure." *Engrg. Fracture Mech.*, 45(2), 245–254.