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Partial safety factor design of rectangular partially prestressed concrete beams in ultimate flexural limit state

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Reliability-based structural design is necessary if uncertainties exist in loads, material and geometric properties and/or mathematical models. The partial safety factors (PSFs) used in reliability-based design, for a class of structural components under a given load combination and failure criteria, should preferably be applicable for a wide range of structural configurations and design options. This paper describes the methodology in detail for developing a set of optimal reliability-based PSFs for given limit state, load combination and target reliability. The class of structural components considered is rectangular partially prestressed concrete beams in ultimate flexure limit state subject to dead and live loads. The mechanical formulation of the flexural limit state is based on the principle behind prestressed concrete design recommended by IS 1343 and SP16. The first order reliability method (FORM) with Rackwitz-Fiessler transformation and gradient projection algorithm are used in this work and the methods are described in detail. Numerical examples involving flexural design of rectangular partially prestressed concrete beams are described. The conservatism in the code specified nominal moment capacity is brought out. A detailed survey of the statistics of related random variables is presented. The variation of the reliability index β as a function of the nominal load ratio, for different values of live load factor, characteristic compressive strength of concrete, nominal prestressing force, percentage reinforcement and eccentricity of the prestressing force, is determined. PSFs optimized for a range of load ratios and nominal prestressing force corresponding to a set of target reliabilities are presented.

KEYWORDS: Structural reliability; prestressed beams; interaction diagram; load and resistance factor design; first order reliability method; ultimate limit state.

Significant uncertainties may be associated with parameters governing the design and construction of a structure-starting with material properties to expected loads to construction methods and models used for analysis. It is most rational to treat such uncertainties in a probabilistic format, specifically, to model the time-invariant quantities as random variables and the time-dependent ones as stochastic processes. Recognizing the existence of these uncertainties is an admission of the fact that the structure may not always satisfy its performance and safety objectives during its intended design life. The logical extension of this admission is to ensure that the likelihood of unsatisfactory performance be kept acceptably low during the life of the structure.

The subject of structural reliability provides the tools and methodologies to explicitly determine the probability of such failures ("failure" here in the sense of non-compliance or nonperformance) by taking into account all relevant uncertainties. These techniques can be used to design new structures with specified (i.e., target) reliabilities, and to maintain existing structures at or above specified reliabilities. Even though such computed probabilities of failure (reliability being 1 minus failure probability) may not have a frequentist or actuarial basis, structural reliability provides a neutral and non-denominational basis to compare different (and often disparate) designs and maintenance strategies on a common basis.

Like any other design approach, reliability based design is an iterative process: the design is adjusted until adequate safety is achieved and cost and functional requirements are met. The final step of meeting the target reliability can either be direct where the computed structural reliability has to exactly satisfy the target reliability for each relevant limit state or it can be indirect as in partial safety factors (PSF) based design where the structure implicitly satisfies the target reliability within a certain tolerance[1]. The term load and resistance factor design (LRFD) implies the approach followed in the United States where the nominal resistance in the design equation is multiplied by an explicit "resistance factor"-but the nominal material properties that go into determining the resistance are not factored. The term PSF based design implies the approach taken in Europe where there is no explicit resistance factor in design, but each material property gene-

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rally has its own partial safety factor. The latter approach is taken in this paper.

This paper determines a set of optimal partial safety factors (PSFs) for the design of partially prestressed concrete rectangular beams in flexural limit state defined by collapse of concrete due to crushing. The problem is formulated as an element reliability problem (as opposed to a system reliability formulation) and concerns the reliability of the most critical section of the beam in flexure. Various material and geometric properties, expected loads etc. are modeled as random variables; the load combination is limited to dead and live loads. First order reliability method (FORM) with gradient projection algorithm is the kernel of the optimization algorithm; a range of target reliabilities in ultimate flexural limit state has been adopted to develop the optimal PSFs corresponding to given frequencies of nominal live to dead load ratios.

The paper begins with a short introduction to the principle behind prestressing of concrete and modes of failure of prestressed concrete beams. The mechanical approach recommended by IS-456[2] to determine capacity of rectangular concrete sections has been described in detail. It is followed by a description of the first order reliability method along with the algorithm for the gradient projection method. The methodology for reliability based load and resistance factor design is described; a brief history of reliability based design and analysis of prestressed concrete beams is presented. Statistics of the basic variables used and the numerical results are given at the end.

DESIGN OF PRESTRESSED BEAMS IN FLEXURE

The tensile strength of concrete is negligible compared to its compressive strength. In ordinary reinforced concrete, the reinforcing steel is used to carry the tensile stresses, and the concrete near the tensile face may crack. Prestressing is intended to artificially induce compressive stresses in the concrete to counteract the tensile stresses caused by external loads, such that the loaded section remains mostly if not entirely in compression[3].

Prestressed sections are therefore more efficiently utilized compared to ordinary reinforced concrete sections. Prestressed concrete members are relatively lightweight as they are built from high strength steel and high strength concrete, more resistant to shear, and can recover from effects of overloading. However, prestressed concrete structures are more expensive, have a smaller margin for error, and the design process of prestressed members is more complicated. Although the loss of prestress with time is built into the design, unintended loss of prestress arising from corrosion of the tendons, slippage etc. can have catastrophic consequences.

Partially prestressed concrete sections are reinforced with prestressed tendons as well as ordinary rebars and can sustain tension under working loads. Partial prestressing occupies the whole spectrum of reinforcement between ordinary reinforced and fully prestressed concrete, and thus constitutes the general case whose two extreme boundaries are represented by fully reinforced and fully prestressed concrete[4].

Prestressed concrete beams may fail in several possible ways (such as a combination of flexure, shear and torsion, bursting of end blocks, bearing, anchorage or connection failures, excessive deflections etc.). This paper however, only looks at ultimate flexural limit state of rectangular beams defined by collapse of concrete due to crushing.

Ultimate moment capacity of prestressed beams

In IS 456[2] the compressive stress-strain relationship for concrete (Figure 1) is taken to be parabolic up to a strain of 0.002, and horizontal from that point on. The design compressive strength of concrete is taken to be f_{ckd} = 0.446 f_{ck} = $f_{ck}/2.25$, where f_{ck} is the characteristic compressive strength of concrete. In essence, 2.25 is the material partial safety factor on concrete strength. The failure strain of concrete in compression is 0.0035. IS 1343[5] specifies minimum grade of concrete as M30 for post-tensioning and M40 for pre-tensioning.

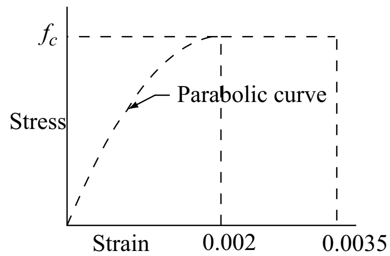

Fig. 1 Stress-strain characteristics of concrete

The design yield stress for reinforcing steel is f_{yd} = $0.87f_{yn}$ = f_{yn}/1.15 where f_{yn} is the characteristic yield strength, and 1.15 is the partial safety factor on steel strength. The modulus of elasticity of steel, E, is 200000 N/mm^2. For mild steel, stress is linearly proportional to strain up to a strain of 0.002 and constant thereafter.

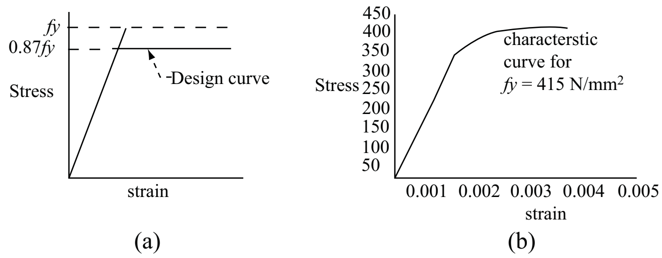

Fig. 2 (a) stress strain characteristic of mild steel (b) high strength steel

For cold-worked (high-strength) bars, the stress-strain relationship is linear-elastic up to a stress of $0.8f_y$ (Fig. 2), after which point inelastic strain starts to develop, as shown in Fig. 2(b). Let s be the total strain in cold-worked steel. The elastic limit for strain is taken to be $s_{pl} = \frac{0.8^* f_{yd}}{E}$. If the total strain is less than s_{pl}, the stress is s^*E. If the total strain s is greater than s_{pl}, the stress can be found by linear interpolation as in Table 1(based on SP16[6]) where $s_{il} = s - s_{pl}$ is the inelastic strain.

TABLE 1
STRESS AS A FUNCTION OF INELASTIC STRAIN BEYOND ELASTIC LIMIT IN COLD WORKED STEEL

Inelastic strain (s_{il})	Stress in steel (f_s)
0-0.0001	$0.8f_y + 0.05\,f_y\,\frac{s_{il}-0}{0.0001-0}$
0.0001-0.0003	$0.8f_y + 0.05\,f_y\,\frac{s_{il}-0.0001}{0.0003-0.001}$
0.0003-0.0007	$0.9f_y + 0.05\,f_y\,\frac{s_{il}-0.0003}{0.0007-0.0003}$
0.0007-0.001	$0.95f_y + 0.025\,f_y\,\frac{s_{il}-0.0007}{0.0001-0.0007}$
0.001-0.002	$0.975f_y + 0.025\,f_y\,\frac{s_{il}-0.001}{0.002-0.001}$
> 0.002	f_y

The parabolic stress block for concrete, and the design procedure adopted in SP-16 are very similar to the rectangular stress block and the design procedure in ACI 318[7]. Naaman[4] undertook a comprehensive study on the behavior of partially prestressed members up to the ultimate point using the nonlinear stress-strain characteristics of steel and concrete. An approximate nonlinear analysis which takes into account the non-linear stress-strain characteristics of steel but considers the ACI assumptions of the equivalent rectangular stress block was also performed. The values of ultimate moment capacity, curvature etc. obtained by the approximate method and the design procedure suggested by ACI were compared with those obtained by the author's nonlinear analysis. It was found that the ultimate moment capacities obtained by ACI are within 7% and on the conservative side of his results. Thus we can conclude that the ACI design procedure and hence the procedure suggested in Indian Standards are sufficiently accurate.

The ultimate moment capacity of a partially prestressed concrete section as a function of prestressing force used in the flexural collapse limit state later in this paper can be obtained in the form of a so-called interaction diagram. Interaction diagrams are plots of normalized compressive force, $P' = \frac{P}{f_{ck}bD}$, and normalized ultimate moment capacity, $M' = \frac{M}{f_{ck}bD^2}$ of a reinforced concrete section, both of which can be expressed as functions of the percentage of steel reinforcement (p) and the location of the neutral axis, $k = \frac{x_u}{D}$. The term x_u is the distance of the neutral axis from the right edge, and b and D are the width and the depth of the section, respectively.

Two cases are possible (Fig. 3) the neutral axis (NA) outside and the neutral axis inside the section. In the former, the entire section is in compression, the maximum strain in concrete (which occurs at the right edge) is between 0.002 and 0.0035, the reinforcements on the right have yielded while those on the left have not. In the latter case, concrete has cracked and thus does not carry any load in the tensile zone, the strain in the most compressed edge of the section is fixed at 0.0035, and the reinforcements on the right have yielded while those on the left may have yielded depending on the value of k.

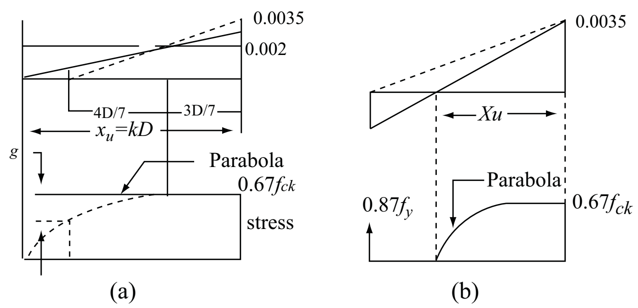

Fig. 3 (a) Strain distribution lines for neutral axis outside and (b) inside the section

For purely axial compression (k = infinity), the strain is assumed to be 0.002 uniformly across the section. As long as the NA lies outside the section, the strain is assumed to be constant at 0.002 at $3D/7$ from the highly compressed edge which acts as the fulcrum for the strain distribution line Fig. 3 (a).

Figure 3 (a) also shows the shape of the stress block for concrete when the neutral axis lies outside the section. The stress is uniform and equal to $0.446 f_{ck}$ up to $3D/7$ from the highly compressed edge and is parabolic for the rest of the section as the strain is less than 0.002.

The expression for g, the difference between the maximum and the minimum stresses in concrete is given as

$$g = f_{ckd}\left(\frac{4}{7k-3}\right)^2 \tag{1}$$

When the neutral axis lies outside the section $(k > 1)$ the normalized compressive force and moment capacity are:

$$P' = \frac{P}{f_{ck}bD} = 0.809\,\frac{f_{ckd}}{f_{ck}}\,C_1 + \sum_{i=1}^{n}\frac{p_i}{100 f_{ck}}\,(f_{si} - f_{ci}) \tag{2}$$

$$M' = \frac{M}{f_{ck}bD^2} = 0.809\,\frac{f_{ckd}}{f_{ck}}\,C_1\,(0.5 - C_2) + P'\,\frac{e}{D} + \sum_{i=1}^{n}\frac{p_i}{100 f_{ck}}\,(f_{si} - f_{ci})\left(\frac{y_i}{D}\right) \tag{3}$$

where,C_1= coefficient for the area of stress block given by:

$$C_1 = 1 - \frac{4}{21}\left(\frac{4}{7k-3}\right)^2 \tag{4}$$

C_2D	=	distance of the centroid of the concrete stress block measured from the highly compressed edge
n	=	number of rows of steel reinforcement
p_i	=	percentage steel reinforcement in the i^{th} row
f_{si}	=	stress in the i^{th} row of reinforcement, compression being positive and tension being
f_{ci}	=	negative stress in concrete at the level of the i^{th} row of reinforcement
y_i	=	distance of the i^{th} row of reinforcement from the centroid of the section, positive for the left edge and negative for the right edge. eccentricity
e	=	of the prestressing force

When the neutral axis lies within the section ($k < 1$), the normalized compressive force and moment capacity are:

$$P' = \frac{P}{f_{ck}bD} = 0.809\,\frac{f_{ckd}}{f_{ck}}\,k + \sum_{i=1}^{n}\frac{P_i}{100 f_{ck}}\,(f_{si} - f_{ci}) \tag{5}$$

$$M' = \frac{M}{f_{ck}bD^2} = 0.809\frac{f_{ckd}}{f_{ck}}\,k\,(0.5 - 0.416k) + P'\,\frac{e}{D} + \sum_{i=1}^{n}\frac{P_i}{100 f_{ck}}\,(f_{si} - f_{ci})\left(\frac{y_i}{D}\right) \tag{6}$$

The expression of the moment capacity given by SP-16 does not include the component $P'\frac{e}{D}$, the moment due to the eccentricity of the prestressing force needs to be included to consider the effect of prestressing on the moment capacity of the section.

Example beam cross-section used in this paper

The rectangular beam section modeled in this paper is considered to have two rows of reinforcement each at a distance d' from the edge as in Figure 4 and each making up half of the reinforcement of the section such that $p_1 = p_2 = p/2$ where p is the total percentage reinforcement and p_1 and p_2 are the percentage of reinforcement for row 1 and row 2 respectively. The figure below the cross-section of the beam shows the force balance with C being the equivalent compressive force and T being the equivalent tensile force. To construct the interaction diagram, f_{s1} and f_{s2}, stresses in reinforcement respectively for the right and left rows and f_{c1} and f_{c2}, the stresses in concrete need to be determined.

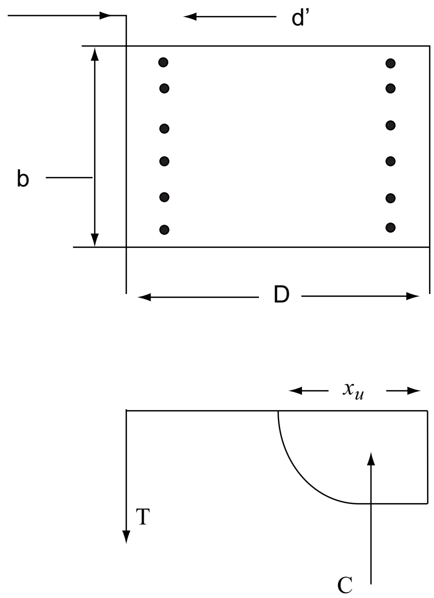

Fig. 4 Rectangular section used in the paper

When $k > 1$, the strains in the right and the left reinforcements are, respectively:

$$s_1 = 0.014 \left(\frac{k - 1 + \frac{d'}{D}}{(7k - 3)} \right) \tag{7}$$

$$s_2 = 0.014 \left(\frac{k - \frac{d'}{D}}{(7k - 3)} \right) \tag{8}$$

The distances of the two rows of reinforcement from the centroid are:

$$\frac{y_1}{D} = (0.5 + d') \tag{9}$$

$$\frac{y_2}{D} = (0.5 - d') \tag{10}$$

when $k < 1$, the corresponding values are:

$$s_1 = 0.0035 \left(\frac{1 - k + \frac{d'}{D}}{k} \right) \tag{11}$$

$$s_2 = 0.0035 \tag{12}$$

$$\frac{y_1}{D} = (0.5 + d') \tag{13}$$

$$\frac{y_2}{D} = (0.5 - d') \tag{14}$$

Figure 5 shows a set of interaction diagrams generated by this method for the following values: f_{ck} = 50 N/mm^2,f_{yn} = 415 N/mm^2, p/f_{ck} = 0 to 0.5%, $e = 0.15D$, $d = 0.05D$.

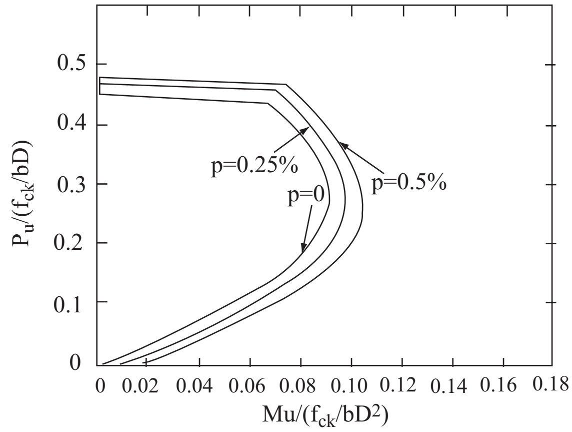

Fig. 5 Interaction diagram

As mentioned above interaction diagrams have been used as a tool to determine the ultimate moment capacity of a concrete section for a given prestressing force. Analytically, for a given P, one would need to determine the position of the neutral axis by iteration using Eq. (2) or (5) depending on the position of the neutral axis, calculate the value of M, using either Eq. (3) or (6) as appropriate. Notice that the graphs drops to $M = 0$ around P' between 0.4 and 0.5 as the section's compressive strength is exceeded at such high prestressing forces.

Reliability based design

Reliability based design, while taking into account various uncertainties associated with a structure, is much like other design processes in revising the design until the demands made of the structure in terms of safety, cost and functions are met. The indirect method of achieving the target reliability such as partial safety factor design and load and resistance factor design aim at a design which implicitly satisfies the target reliability within a certain tolerance. The method used in this paper to determine the reliability of a structural component and the procedure of obtaining the optimal load and resistance factors to be used in design is described below.

First Order Reliability Method

A limit state or performance function, $g(\underline{X})$,for a structural component is defined in terms of the basic variables, $\underline{X}$, such that $g(\underline{X}) < 0$ denotes failure, $g(\underline{X}) > 0$ denotes satisfactory performance, and the surface given by $g(\underline{X}) = 0$ is called the limit state equation or limit state surface. The performance function g is typically obtained from the mechanics of the problem at hand.

The basic variables generally comprise of quantities like material properties, loads or load-effects, environmental parameters, geometric quantities, modeling uncertainties, etc. They are usually modeled as random variables; however, those with negligible uncertainties may be treated as deterministic. The general expression of failure probability is

$$P_f = P(g(x) < 0) = \int_{g(\underline{x})<0} f_{\underline{X}}(\underline{x}) d\underline{x} \tag{15}$$

where $f_X(x)$ is the joint probability density function for $\underline{X}$. The reliability of the structure would then be defined as $Rel = 1 - P_f$.

Closed-form solutions to Eq. (15) are generally unavailable. Two different approaches are widely in use: (i) analytic methods based on constrained optimization and normal probability approximations, and (ii) simulation based algorithms with or without variation reduction techniques and both can provide accurate and efficient solutions to the structural reliability problem. The first kind, grouped under First Order Reliability Methods (or FORM), holds a distinct advantage over the simulation based methods in that the design point(s) and the sensitivity of each basic variable can be explicitly determined.

FORM calculates the reliability of a system by mapping the failure surface onto the standard normal space and then by approximating it with a tangent hyperplane at the design point (defined as the point on the limit state surface in the standard normal space that is closest to the origin)[8]. Provided the reliability problem is well-behaved and straightforward, the solutions obtained by FORM are reasonably close to that obtained by the relatively expensive simulation based solutions. FORM has been used in this paper to determine the reliability of the prestressed designs being considered.

The two important steps of FORM are described in detail in the following.

1. Map the basic variables $\underline{X}$ on to the independent standard normal space $\underline{Y}$ and hence $g(\underline{X})$ to $g_1(\underline{Y})$. Several mappings are possible, such as (i) Hasofer-Lind[9] or second moment transformation which uses information only on the first two moments of each X, (ii) Nataf transformation[10] which uses marginal distribution of each X and the correlation matrix of the $\underline{X}$ vector, (iii) Rosenblatt transformation[10] which uses nth order joint distribution information, a special case of which is the so-called full distribution transformation valid when the $\underline{X}$ are mutually independent. This paper uses the Rackwitz-Fiessler[11] transformation which converts each X point-by-point into an equivalent normal U through a marginal distribution and density equivalence, and then the vector $\underline{U}$ into the independent standard normal vector $\underline{Y}$ through a Nataf type transformation. At any candidate checking point, $\underline{x}^*$, the standard deviation and mean of the equivalent intermediate normal, U_i, are:

$$\begin{aligned} {\sigma_i}^N &= \frac{\phi\left(\Phi^{-1}\left(F_i\left(x_i^*\right)\right)\right)}{f_i\ \left(x_i^*\right)} \\ {\mu_i}^N &= x_i^*\ - \Phi^{-1}\ \left(F_i\left(x_i^*\right)\right){\sigma_i}^N \end{aligned} \quad (16)$$

F_i and f_i are the cumulative distribution function and the probability density function of the original X_i, respectively. The intermediate $\underline{U}$ vector is generally dependent, and is mapped onto the space of independent standard normals, $\underline{Y}$, through the following transformation:

$$\underset{\sim}{y} \ = \ L^{-1}\ \underset{\sim}{z} \ \text{ where } z_i \ = \ \frac{u_i - {\mu_i}^N}{{\sigma_i}^N} \quad (17)$$

L is the lower triangular matrix obtained by Cholesky factorization of the correlation matrix, R', of $\underline{Z}$, which deviates from the correlation matrix R, of $\underline{X}$, on account of the nonlinear transformation between each X_i and Z_i, but the deviation is slight and can be easily corrected[12].

2. Locate on g_1 the point $\underset{\sim}{y}$* closest to the origin,

$$\begin{aligned} &\min\ F = \underset{\sim}{y}^T \underset{\sim}{y} \\ &\text{subject to}\ G = g_1\left(\underset{\sim}{y}\right) = 0 \end{aligned} \quad (18)$$

Let the solution to this optimization problem be $\underline{y}^*$ and let β be the distance of this optimal point from the origin. This minimum norm point $\underline{y}^*$, is known as the checking or the design point. The limit state surface $g1$ can be approximated by a tangent hyperplane at y^*, yielding the approximate probability of failure as

$$P_f \ = \ \Phi\ (-\beta\, \mathrm{sgn}\ [g_1\ (\underline{0})]) \quad (19)$$

The signum function determines whether the origin is in the safe domain or not. The drawback of FORM is that it provides the exact solution only if the original limit state is linear and the basic variables are normally distributed. Otherwise, the extent of error depends on the curvature of the limit state and the method of mapping of $\underline{X}$ onto $\underline{Y}$.

After performing a FORM analysis, the design point $\underline{y}^*$ can be transformed back into the basic variable space, yielding the "checking point", $\underline{x}^*$ which cannot be obtained from simulation based solutions. It is implied that if the structural element in question is designed using this combination $\underline{x}^*$, the reliability of the component would be β (within the approximations of FORM). This, in fact is the basis of partial safety factor design, discussed subsequently.

The gradient projection method, originally developed by Rosen[13], is well-suited to tackle the constrained non-linear optimization problem in Eq. (18) and has been adopted in this paper. The essential steps of the gradient projection method as applied to FORM are described next[13].

Gradient projection method

The gradient projection method is a modified version of the steepest descent method for unconstrained optimization. To solve the optimization problem(18), one can determine a direction of search d_k from the current point y_k and then search for a new point along that direction as

$$y_{k+1} = y_k \ + \ \alpha_k\, d_k \quad (20)$$

The new point is the optimal point along d_k, however it may not be the optimal point of the entire feasible set. Hence, at every new point this process is repeated until the point satisfies the optimality conditions. The process of searching for the minimum point along a direction is called a line search. Armijo's rule has been used for the line search in this study which states that the scalar step size α_k is acceptable (neither too large nor too small) if it satisfies the following conditions:

$$F\left(y_k + \ \alpha_k d_k\right) \le F\left(y_k\right) + \varepsilon\alpha_k \nabla\, F\ \left(y_k\right)\ d_k \quad (21)$$

$$F\left(y_k + \eta\alpha_k d_k\right) \succ F\left(y_k\right) + \varepsilon\eta\alpha_k \nabla\, F\left(y_k\right)\ d_k \quad (22)$$

Values 0 < ε < 1 and η > 1, and ε = 0.2 and η = 2 are often used.

The new direction of search d_k is taken to be the projection of the negative gradient of the objective function onto the tangent plane of the feasible set since each point must remain in the feasible domain:

$$d_k = -\left[I - \frac{\nabla G\left(y_k\right)^T \nabla G\ \left(y_k\right)}{\left|\nabla G\left(y_k\right)\right|^2}\right]\ y_k \quad (23)$$

The entire algorithm is described in Fig. 6.

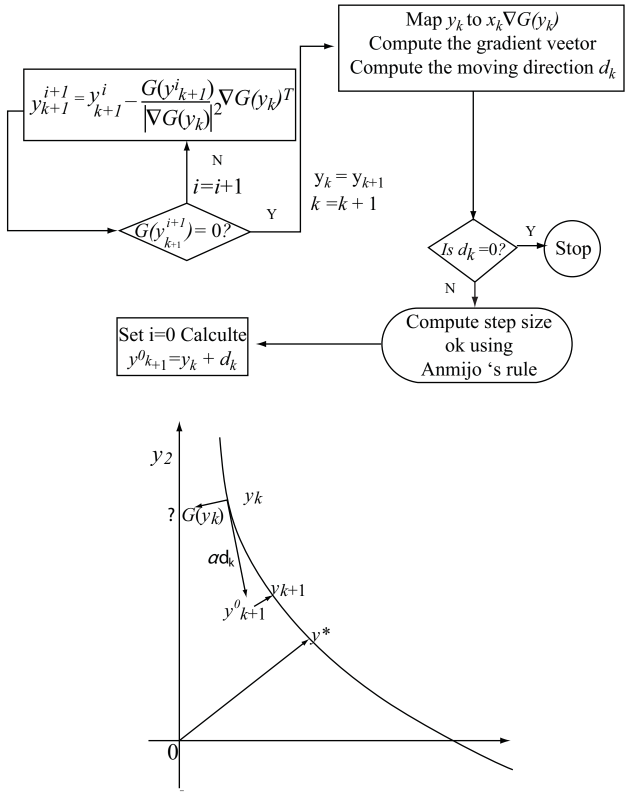

Fig. 6 Gradient projection method

Reliability based load and resistance factor design

Reliability based load and resistance factors in a so-called level 1 safety checking design format are intended to ensure a nearly uniform level of reliability across a given category of structural components for a given class of limit state under a particular load combination[14]. The design or checking point, X^*, obtained from a FORM analysis, satisfies

$$g\left(\underline{X}^*\right) = 0 \tag{24}$$

Since nominal or characteristic values of basic variables, instead of checking point values, are typically used in design, Eq. (24) may be written as:

$$g\left(\frac{X_1^n}{\gamma_1}, \ldots, \frac{X_k^n}{\gamma_k}, \gamma_{k+1} X_{k+1}^n, \gamma_{k+2} X_{k+2}^n, \cdots, \gamma_m X_m^n\right) \geq 0 \tag{25}$$

where the superscript n indicates the nominal value of the variable. We have partitioned the vector of basic variables into k resistance type and $m-k$ action type quantities. The partial safety factors, γ_i, are typically chosen to be greater than unity, such that they divide the nominal resistance values and multiply the nominal action values to yield the design values:

$$\text{resistance type PSFs}: \ \gamma_i = \frac{X_i^n}{X_i^*} i = 1, \ldots, k \tag{26}$$

$$\text{action type PSFs} \ \gamma_i = \frac{X_i^*}{X_i^n} i = k+1, \ldots, m \tag{27}$$

If the checking point equation can be separated into a strength term and a load-effect term due to simulataneously acting loads, the following safety checking scheme may be adopted for design:

$$R_n\left(\frac{X_i{}^n}{\gamma_i}, \ i = 1, \ldots, k\right) \geq l\left(\sum_{i-1}^{m-k} \gamma_i Q_{ni}\right) \tag{28}$$

where R_n is the nominal resistance function of the member being considered, and l is the load effect function, and Q_{ni} represents the nominal value of the i^{th} load. Note that there is no separate resistance factor (ϕ) multiplying the nominal resistance (as in LRFD) since material partial safety factors have already been incorporated in computing the strength. As stated above, the partial safety factors (PSFs) are optimized in some sense to be valid for a class of structural components in some given limit state and load combination.

Although the FORM approach gives a lucid explanation of how and why of the load and resistance factors, their optimal values do not necessarily have to involve FORM analyses; simulation-based analyses are equally adequate. The essential steps in developing optimal PSFs are described in Fig. 7.

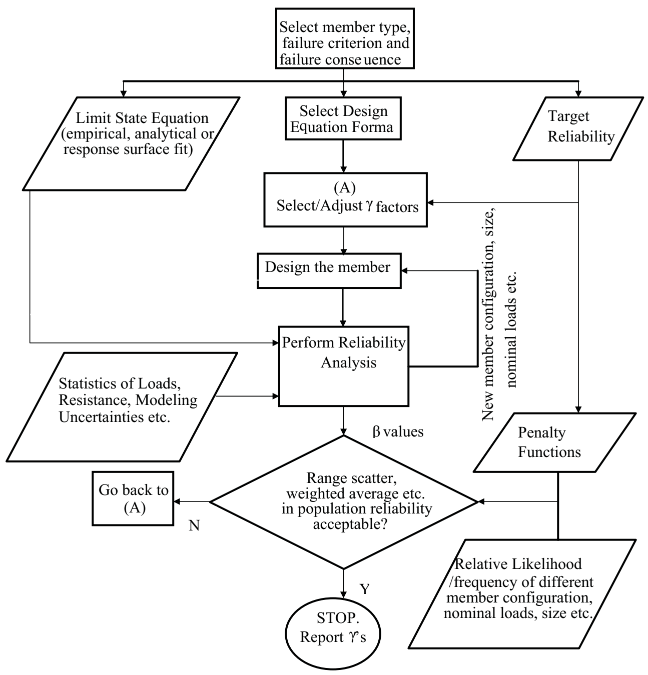

Fig. 7 Determination of load and resistance factors

Of particular interest is the optimality criteria and constraints in determining the PSFs (or LRFs in LRFD). Let there be n representative structural components selected to develop the optimal factors, and let w_i be the weight (or relative importance, or relative frequency) assigned to the i^{th} such component. For a given set of PSFs, the reliability of the i^{th} component is β_i. Let β_T be the target reliability index (equivalent to a target reliability of $\Phi(\beta_T)$) for the components in the given limit state.

The optimality criteria may be as simple as:

$$\min \beta_i\left(\gamma_k, k = 1, \ldots, m\right) \geq \beta_T \tag{29}$$

which, however, would lead to mostly overdesigned components. An alternative approach could be to minimize the sum of weighted squared deviations from the target:

$$\min \left[\sum_{i=1}^{n} W_i \left(\beta_i \, (\gamma_k, \, k=1,...,m) - \beta_T\right)^2\right]$$
$$\text{where} \sum_{i=1}^{n} W_i \, = \, 1 \tag{30}$$

Hansen and Ditlevsen[15] provide an alternative approach:

$$\min \sum\nolimits_i \left[c\left(\beta_i\left(\gamma,\varphi\right) - \beta_T\right) + \exp\left(-c\left(\beta_i\left(\gamma,\varphi\right) - \beta_T\right)\right) - 1\right] \tag{31}$$

and recommended a value of $c = 4.35$ for good results.

NBS Spl. Pub. 577[16] adopted the weighted squared difference between the nominal resistance and required nominal resistance for each component as the objective to be minimized. The former is the result of the design equation with trial LRFs. The latter is the nominal resistance required to exactly satisfy the target reliability for the given structure. Working on reliability based design of nuclear power plant containments, Hwang et al.[16] took the weighted squared deviation of the log-failure probabilities as the objective. More than a decade later, while calibrating the ultimate limit state design of mooring lines, Horte el al.[17] took sum of the squared deviations of the failure probabilities from the target value as the objective as it provided "a high penalty for under-designed cases".

In addition to the objective function, several constraints may be introduced to satisfy engineering and policy considerations, such as:

$$\begin{aligned} &\beta_T \, - \, \min \, \beta_i \, \left(\gamma,\varphi\right) \, \leq \, \Delta_{\max} \\ &\varphi_i \leq 1 \text{ for some or all } i \\ &\gamma_j \geq 1 \text{ for some or all } j \end{aligned} \tag{32}$$

etc.

where Δ_{max} is the maximum permissible deviation below the target.

RELIABILITY ANALYSIS OF PRESTRESSED BEAMS

Al-Harthy and Frangopol[18] performed a comprehensive study on 73 prestressed beams designed to the 1989 ACI 318 standard to find out their implied reliability levels. Three types of limit states were considered: (i) ultimate strength in flexure, (ii) cracking in flexure, and (iii) permissible stresses at initial and final stages of prestressing (due to both loading and prestressing). Only dead and live load effects were considered. The random variables included dead and live loads, material properties (concrete tensile and compressive strengths, concrete density), geometric properties (area of prestress strands, beam dimensions), initial, final and ultimate prestressing forces. Modeling uncertainty in estimating the "behavior of prestressed concrete beams" was also incorporated as a random variable, but the authors have not elaborated on the formulation. All random variables were taken to be normally distributed except live load (taken to be Type I maximum). The effect of correlation among some of the random variables was also studied. The Rackwitz-Fiessler[11] algorithm was used in the FORM analysis. The authors concluded that the reliability indices implied by the 1989 ACI 318 design standard are non-uniform over various ranges of loads, span lengths and limit states. The limit state of permissible tension in the final stage was found to be critical in most cases. They recommend the next logical step as the determination of target reliability and developing consistent load and resistance factors for design.

Hamann and Bulleit[19] took a slightly different approach to examining the reliability of under-reinforced high-strength concrete prestressed beams designed in accordance with the 1983 ACI-318 standard. They looked only at the ultimate flexural limit state of beams subjected to dead and snow loads. Where Al-Harthy and Frangopol included all the material and geometric random variables in the FORM analysis, Hamann and Bulleit first estimated the moment capacity through Monte Carlo simulations, fitted the data to standard distributions, and then performed a first order second moment reliability analysis on the linear limit state. M_{cap} was determined, as a function of the material and geometric properties, through an iterative non-linear analysis which included the shape of the stress-strain curves of concrete and steel as given by Naaman[4].

Reliability for Class-1 structures, particularly concrete containment structures for nuclear power plants, is a much researched subject primarily due to the dire failure consequences of the containment structure in terms of environmental impact, human casualties and other economic costs. Hwang et al.[16] described a Load and Resistance Factor Design (LRFD)-based approach to determine the critical load combinations for design of concrete containment structures. The limit state, corresponding to ultimate strength of concrete, was defined in the 2-D space of membrane stress and bending moment in the shell, leading to an octagonal limit state surface. Yielding of reinforcements was permitted. Working also on the reliability of concrete containments, Pandey[20] on the other hand took the limit state as tensile cracking of concrete to represent the failure mode of through-thickness cracking. Based on a set of flexural, lift-off and destructive tests over time on 16 representative beams, he proposed a more quantitative approach to update the probability distributions of the prestressing force and the number of degraded tendons in the containments. Varpasuo[21] focused on seismic reliability of a VVER-1000 containment structure and hence took cracking of concrete after yielding of reinforcement as the limit state. Both Pandey's and Varpasuo's limit states form sides of the octagonal limit state considered by Hwang et al along with failure corresponding to simultaneous yielding of reinforcement and cracking of concrete.

NUMERICAL RESULTS

Limit state and design equation

We consider only flexural limit state for the beam under dead and live loads. Failure is defined as crushing of concrete while reinforcements are allowed to yield. The mechanics of this failure criterion has been described in detail in Section 2. The limit state equation is:

$$M_{cap} \; - \; (M_{DL} \; + \; M_{LL}) \; = \; 0 \tag{33}$$

where M_{cap} = Moment capacity, M_D = Moment due to dead load, M_L = Moment due to live load. The corresponding PSF format is:

$$M_{cap_{,n}} \; > \; \gamma_D \, M_{Dn} \; + \; \gamma_L \, M_{Ln} \tag{34}$$

where γ_D = dead load factor, γ_L = live load factor, $M_{cap,n}$ = nominal moment capacity, M_{Dn} = nominal dead load moment, M_{Ln} = nominal live load moment. Note that $M_{cap,n}$ already includes the effect of material parital safety factors on concrete and steel strengths. The objective here is to obtain optimized PSFs for a range of structural configurations and design options corresponding to a target reliability, β_T. As we will see next, within the scope of the above limit state and design equations, the structural configuration is completely specified by the ratio of the two nominal loads, while the design options are specified by the choice of concrete strength, steel yield strength, eccentricity of the prestressing force and percentage of reinforcements.

As described before, the normalized moment capacity,

$$M'_{cap} = \frac{M_{cap}}{f_{ck} b D^2} = M'_{cap}\ (P', f_c, f_y, E, p/f_{ck}, P'_n, e) \quad (35)$$

is a function of the prestressing force P', compressive strength of concrete f_c, yield strength of steel f_y, Young's modulus of reinforcing steel E, eccentricity of the prestressing force e and the normalized percentage reinforcement p/f_{ck} for a given section. Likewise, the nominal moment capacity is:

$$M'_{cap,n} = \frac{M_{cap,n}}{f_{ck} b D^2} = M'_{cap,n}\ (P'_n, f_{ckd}, f_{yd}, E_n, p/f_{ck}, e) \quad (36)$$

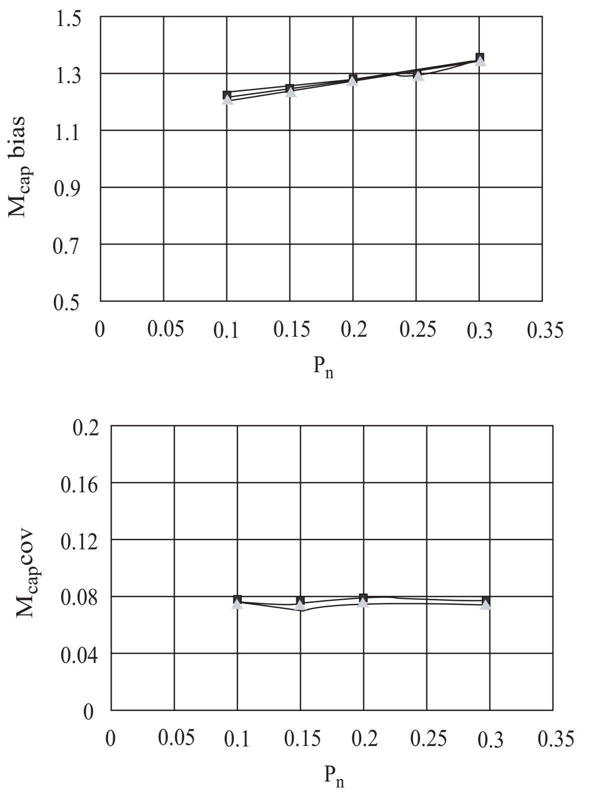

Fig. 8 Bias and c.o.v. of Mcap respectively as a function of P_n for three values of f_{ck} (40, 50, 60 Mpa)

Since the moment capacity is an implicit function of four basic variables, its normalized distribution is not readily available, unlike the two normalized loads whose distribution functions are well-documented. For each call to evaluate the limit state, the random as well as the nominal moment capacities are separately obtained from the interaction diagram. Eq. (3) in conjunction with Eq. (3) or Eq. (6) with Eq. (5) depending on the location of the NA are used to compute the, nominal moment capacity. The same set of equations is used to compute the random moment capacity with f_{ckd} replaced by f_c and f_{yd} replaced by f_y. Note that the design material properties $f_{ckd} = f_{ck}/2.25$ and $f_{yd} = f_{yn}/1.15$ already include the partial safety factors of 2.25 and 1.15 respectively on the nominal values.

Figure 8 shows the variation of the bias (mean/nominal) and c.o.v. (std. dev./mean) of the moment capacity as a function of nominal prestressing force for three different values of characteristic strength of concrete. Other parameters are fixed at $e/D = 0.15, d'/D = 0.05, p = 0.2\%$.

Clearly, there is substantial conservatism in the code-specified nominal strength-the bias can be as high as 1.4 depending on the level of the prestressing force. Such high bias is partly the result of the two factors of safety (each equal to 1.5) employed successively on the characteristic strength of concrete at the design stage (i.e., $f_{ckd} = 0.446 f_{ck} = f_{ck}/1.5/1.5$)-a practice whose rationality may be questioned.

Optimized load and resistance factors

The limit state equation can be normalized with the design equation:

$$g_n = \frac{M_{cap}}{M_{cap,n}} - \frac{M_D/M_{Dn} + (M_L/M_{Ln})(M_{Ln}/M_{Dn})}{\gamma_D + \gamma_L\ (M_{Ln}/M_{Dn})} = 0 \quad (37)$$

If the statistics of the three normalized random variables above (capacity, dead load and live load) are known and the choice of steel and concrete are given, the reliability index for the above limit state equation would depend on the LRFs, the nominal load ratios and the percentage reinforcement:

$$\beta = \Phi^{-1}\ (g_n > 0) = \beta\ (\gamma_D, \gamma_L, M_{Ln}/M_{Dn}, p, P'_n; f_{ck}, f_{yn}, e) \quad (38)$$

If there are n_r different nominal load ratios $r_i (M_{Ln}/M_{Dn})_i$ with weights $w^r{}_i$, and n_p choices of nominal prestressing force P_{nj}, with weights w_i^p, the optimal LRFs are the solution of the following problem:

$$\min \left[\sum_{j=1}^{n_p} \sum_{i=1}^{n_r} w_i^r\, w_j^p\ (\beta(\gamma_D, \gamma_L; \gamma_i, P_{nj}) - \beta_T)^2 \right]$$

$$\text{where } \sum_{i=1}^{n_r} w_i{}^r = 1,\ \sum_{j=1}^{n_p} w_i{}^p = 1 \quad (39)$$

subject to:

$$\gamma_D > 1$$
$$\gamma_L > 1$$

The distribution and statistics of the basic variables are listed in Table 2. All basic variables are assumed to be mutually independent. These have been extracted from a larger set collected from the available literature as described in the Appendix. The deterministic parameters and their values used are listed in Table 3.

TABLE 2 STATISTICS OF BASIC VARIABLES		
Random Variable	Description	Statistics Distribution (mean, c.o.v.)
P'	normalized prestressing force	Lognormal ($1.15P_n$,10%)
f_c	compressive strength of concrete	Normal$(f_{ck} + 0.825s_c, s_c)^*$
f_y	Yield strength of steel	Lognormal(1.1133 f_{yn},0.09)
E	Young's modulus	Normal(1.001103 E_n,0.01)
M_D/M_{Dn}	Normalized dead load moment	Normal(1,0.1)
M_L/M_{Ln}	Normalized live load moment	Type 1(0.9,0.3)
s_c = standard deviation for characteristic strength of concrete as given in IS 1343[5]		

The weights for nominal live to nominal dead load moments (Table 4) have been taken from NBS SP577[16]. Strictly speaking these weights are for reinforced concrete structures, but are assumed to be applicable to prestressed beams. The parameter P_n is taken to have 3 values, 0.15, 0.2 and 0.25 having a weight distribution o as shown in Table 5. Three different values for the target reliability index have been taken: 3.0, 3.5 and 4.0.

TABLE 3 VALUES OF DETERMINISTIC PARAMETERS		
Parameter	Description	Values taken
M_{Ln}/M_{Dn}	Nominal live to nominal dead load moment ratio	0.25, 0.5, 1.0, 1.5, 2
p	Percent reinforcement	0.2%
f_{ck}	Characteristic compressive strength of concrete	40, 50 and 60 MPa
f_{yn}	Nominal yield strength of reinforcing steel	415 MPa
E_n	Nominal YoungŠs modulus of reinforcing steel	200 GPa
$P_n = P_n/(f_{ck}bD)$	Normalized prestressing force	0.15, 0.2 and 0.25
β_T	Target reliability index	3.0, 3.5, 4.0
e/D	Eccentricity of prestressing force	0.15

TABLE 4 RELATIVE WEIGHTS FOR NOMINAL LOAD RATIOS					
M_{Ln}/M_{Dn}	0.25	0.5	1.0	1.5	2.0
weight	0.1	0.45	0.3	0.1	0.05

TABLE 5 RELATIVE WEIGHTS FOR NOMINAL PRESTRESSING FORCE			
P_n	0.15	0.2	0.25
weight	0.3	0.4	0.3

We first look at the variation of the reliability index Eq. (38) as a function of the nominal load ratios. Two of the parameters $\gamma_D = 1.1$ and $f_{yn} = 415$MPa are kept fixed. The other five, $\gamma_L, p, f_{ck}, P_n'$ and e/D are varied one at a time, as shown in Figs. 9 to 13 respectively.

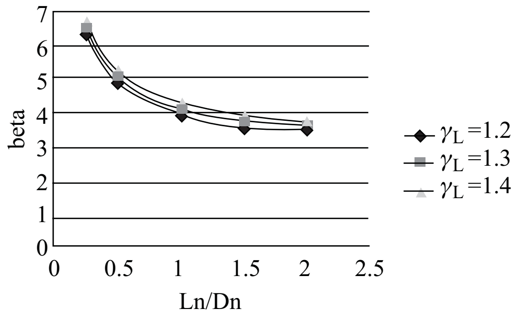

Fig. 9 beta vs. Ln/Dn with varying γ_L and fixed p = 0.2%, P_n = 0.2,eby D = 0.15 and f_{ck} = 50MPa

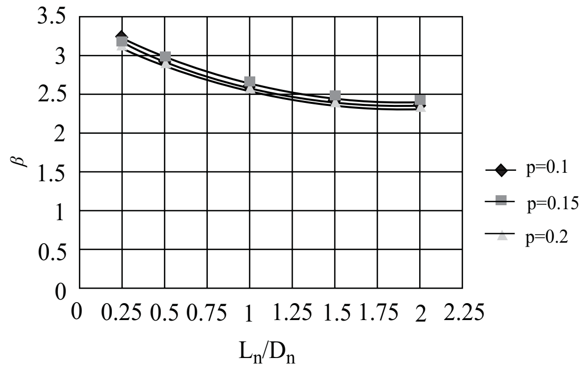

Fig. 10 beta vs. Ln/Dn with varying p and fixed γ_L = 1.3, P_n = 0.2,eby D = 0.15 and f_{ck} = 50MPa

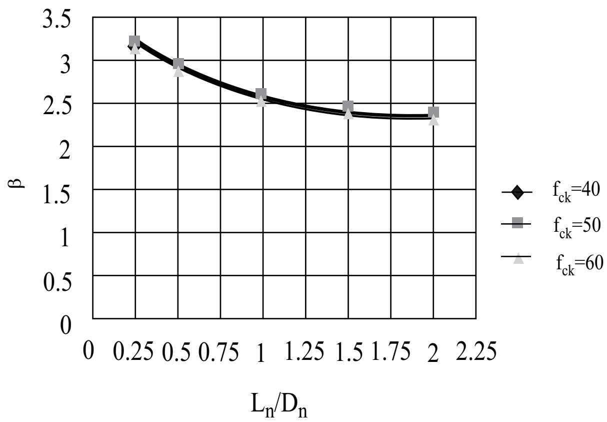

Fig. 11 beta vs. Ln/Dn with varying f_{ck} and fixed p = 0.2% and γ_L = 1.3, P_n = 0.2,eby D = 0.15

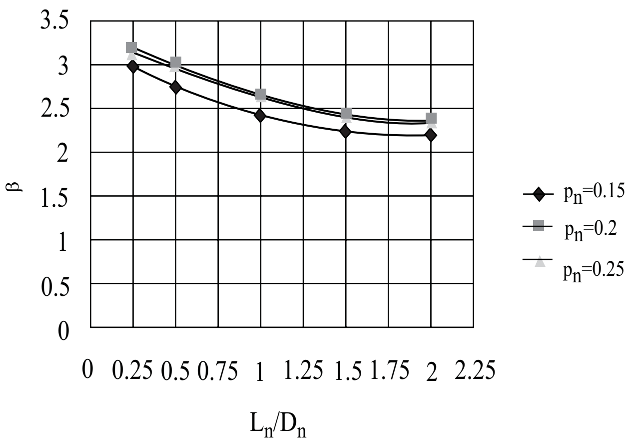

Fig. 12 beta vs. Ln/Dn with varying p_z and fixed p_n = 0.2% and γ_L = 1.3, f_{ck} = 50Mpa, $e/D = 0.15$

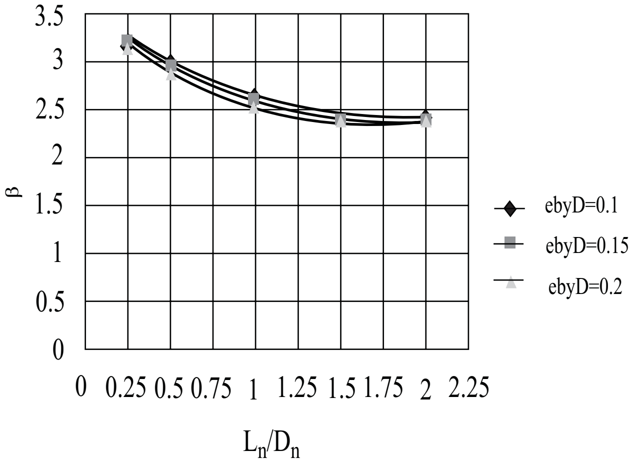

Fig. 13 beta vs. Ln/Dn with varying e/D and fixed p = 0.2% and γ_L = 1.3, P_n = 0.2, f_{ck} = 50Mpa

As can be seen from the figures the reliability decreases with increasing load ratio. Further, at any given load ratio, reliability increases as expected with increasing live load factor, or increasing nominal prestressing force, or increasing concrete strength or increasing eccentricity. The effect of increasing p, f_{ck} and e/D is almost insignificant leading us to conclude that the contribution of concrete strength and non prestressed steel reinforcement to flexural reliability is relatively small.

Finally the optimized PSFs for various combinations of β_T and f_{ck} are shown in Table 6. Of course, desirable properties like low shrinkage, low creep characteristics and high tensile strength may automatically lead to the choice of a higher grade of concrete.

TABLE 6
OPTIMAL LOAD AND RESISTANCE FACTORS (γ_D, γ_L) FOR DIFFERENT COMBINATIONS OF β_T AND f_{ck}

		Target reliability index, β_T		
		3.0	3.5	4.0
Characteristic strength of concrete, f_{ck}	40 MPa	1, 1.54	1, 1.87	1.12, 2
	50 MPa	1, 1.55	1, 1.88	1.12, 2
	60 MPa	1, 1.56	1, 1.87	1.1, 2
Recommended LRFs for all concrete grade		1, 1.55	1, 1.88	1.1, 2

CONCLUSIONS

This paper described the detailed reliability based methodology for developing an optimal set of partial safety factors. The formulation centered on rectangular partially prestressed beams in flexure subject to dead and live loads with failure defined by crushing of concrete. The mechanistic formulation was based on IS1343, IS456 and SP16. Work is in progress to include more load combinations, realistic load ratios and their weights, loss of prestressing force and modeling uncertainties, and to look at serviceability limit state of class I prestressed structures.

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APPENDIX

AVAILABLE STATISTICS OF RELEVANT BASIC VARIABLES FROM THE LITERATURE

Variable	Distribution type	mean relationship	COV	Context/Source
f_{ck}	normal	0.67f'cn - 1.17f'cn	0.1-0.25	Prestressed concrete beams(a)
	normal	1.03f'cn + 5.2(MPa)	0.14	Nuclear containment structure(b)
	normal	1.02f'cn + 1219(ksi)	0.14	Nuclear containment structure(c)
	normal	46.3Mpa	0.106	Nuclear containment structure(d)
	normal	0.895f'cn	0.15	Prestressed conc. beams(e)
	normal	0.675f'cn + 1100(psi)	0.15	Prestressed conc. beams(f)
Fy	lognormal	71.0ksi	0.11	(b)
	normal	530.5Mpa	0.036	(c)
	normal	1.1133fyn	0.09	(e)
Fpu	normal	1.0387fpun	0.0142	(a)
		1820MN/m^2		(c)
	normal	1.0387fpun	0.0142	(e)
	normal	281ksi	0.025	(f)
Aps	normal	1.01176Apsn	0.0125	(a)
	normal	0.1548sq.in	0.0125	(f)
Eps	normal	1.011Epsn	0.01	(a)
	normal	1.01103Epsn	0.01	(e)
H	normal	hn	0.04	(a)
	normal	hn	1/4hn	(f)
B	normal	bn	0.045	(a)
	normal	bn+5/32	1/4(bn+5/32)	(f)
Dead load(D)	normal	Dn	0.1	(a)
	normal	Dn	0.1	(e)
	normal	Dn	0.1	(f)
M_D	normal	nominal	0.07	(d)
	normal	1.05MDn	0.075	(g)(Prestressed conc. girders)
density of concrete(d_c)	normal	d_{cn}	0.1	(e)
	normal	150(lb/cu.ft)	0.1	(f)
F_{py}	normal	1.027fpyn	0.022	(a)
Live Load(L)	type 1	0.894	0.25	(e)
	type 1	0.894	0.25	(f)
$\alpha1$	normal	$\alpha1$n	0.03	(e)
	normal	0.945	0.03	(f)
$\alpha2$	normal	$\alpha2$n	0.0043	(e)
	normal	1.01	0.0043	(f)

(a)-Hamann&Bulleit[19]
(b)-Hwang et al[22]
(c)-Varpasuo[21]
(d)-Pandey[20]
(e)-Barakat et al[23]
(f)-Al-Harthy and Frangopol[24]
(g)Nowak[25]

REFERENCES

1. Bhattacharya, B., R. Basu, and K.-t. Ma, *Developing target reliability for novel structures: the case of the Mobile Offshore Base.* Marine Structures, 14(12), 2001, pp 37–58.

2. BIS, IS456:2000 *Indian Standard Plain and Reinforced Concrete-Code of Practice (fourth revision)*. 2000, Bureau of Indian Standards, New Delhi.

3. Raju, N.K., *Prestressed Concrete, 4th Edition*. 2007, New Delhi: Tata McGraw Hill.

4. Naaman, A.E., *An approximate nonlinear design procedure for partially prestressed concrete beams*. *Comp. and Structs.*, 17(2), 1983, pp 287–299.

5. BIS, IS 1343 *Code of Practice for Prestressed Concrete*. 2003, Bureau of Indian Standards, New Delhi.

6. BIS, SP-16 *Design Aids for Reinforced Concrete to IS:456-1978 Eleventh Reprint*. 1999, Bureau of Indian Standards, New Delhi.

7. ACI-318, *Building Code Requirements for Reinforced Concrete*, *ACI Standard* 318-89. 1989, American Concrete Institute, Detroit, Michigan.

8. Shinozuka, M., *Basic analysis of structural safety. Jl. of Struct. Engg., ASCE*, 109(3), 1983, pp 721-s-

9. Hasofer, A.M. and N.C. Lind, *Exact and invariant second-moment code format*. *Jl. of Engg. Mech., ASCE*, 100(1), 1974, pp 111–121.

10. Melchers, R.E., *Structural Reliability Analysis and Prediction.* 1987, Chricester, UK: Ellis Horwood.

11. Rackwitz, R. and B. Fiessler, *Structural reliability under combined random load sequences. Comp. and Structs.*, 9, 1978, pp 489–494.

12. der Kiureghian, A. and P.-L. Liu, *Structural reliability under incomplete probability information*. *Jl. of Engg. Mech., ASCE*, 112(1), 1986, pp 85–104.

13. Rosen, J.B., *The gradient projection method for nonlilnear programming. Part II. Nonlinear constraints. Jl. of the Society for Industrial and Applied Mathematics* 9(4), 1961, pp 514–532.

14. Ellingwood, B.R., LRFD: *implementing structural reliability in professional practice*. *Engg. Structs.* 22, 2000, pp 106–115.

15. Hansen, P.F. and O. Ditlevsen, *Methodology for Calibration of Partial safety Factors, in Final Report*. 1997.

16. H.Hwang, S.K., M.Reich, B.Ellingwood, M. Shinozuka, *Probability-based load combinations for the design of concrete containments*. Nuclear Engineering and Design, 86, 1985 pp 327–339.

17. Horte, T., H. Lie, and J. Mathisen. *Calibration of an ultimate limit state for mooring lines. in 17th Intl. Conf. on Offshore Mech. and Arctic Engg.* 1998: ASME.

18. Al-Harthy, A.S. and D.M. Frangopol, *Reliability assessement of prestressed concrete beams. J. of Struct. Engg.*, 120(1), 1994.

19. Hamann, R.A. and W.M. Bulleit. *Reliability of prestressed high-strength concrete beams in flexure. in Fifth International Conference on Application of Statistics and Probability in Soil and Structural Engineering*, 1987, Vancouver.

20. Pandey, M.D., *Reliability-based assessment of integrity of bonded prestressed concrete containment structures*. Nuclear Engineering and Design, 176, 1997, pp 247–260.

21. Varpasuo, P., *The seismic reliability of VVER-1000 NPP prestressed containment building.* Nuclear Engineering and Design, 160, 1996, pp 387–398.

22. Hwang, H., et al., *Probability-based load combinations for the design of concrete containments.* Nuclear Engineering and Design, 86 1985, pp 327–339.

23. Barakat, S., K. Bani-Hani, and M.Q. Taha, *Multi-objective reliability-based optimization of prestressed concrete beams.* Structural Safety, 26, 2004, pp 311–342.

24. Al-Harthy, A.S. and D.M. Frangopol, *Integrating system reliability and optimization in prestressed concrete design. Comp. and Structs.*, 64(1-4), 1997, pp 729–735.

25. Nowak, A.S., *Calibration of LRFD bridge code, Jl. of Struct. Engg., ASCE*, 121(8), 1995, pp 1245–1251.