Common continuous probability distributions
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Distribution (explanation)	**PDF**	**CDF**	**Relation between parameters and moments**
Uniform	$f_X(x) = \begin{cases} 1/(b-a), & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$	Linearly increases from 0 at *a* to 1 at *b*	μ_X = (b+a)/2 $\sigma_X^2 = (b-a)^2/12$
Exponential (Weibull with x_0=0, k=1) (Gamma with k=1)	$f_X(x) = \lambda e^{-\lambda x}, \quad x \ge 0$	$F_X(x) = 1 - e^{-\lambda x}, \qquad x \ge 0$	$\lambda = 1/\mu_X$ $\sigma_X = 1/\lambda$
Rayleigh (Weibull with x_0=0, k=2) (Chi with 2 dofs)	$f_X(x) = \frac{2}{u}\left(\frac{x}{u}\right) \exp\left(-\left(\frac{x}{u}\right)^2\right)$	$F_X(x) = 1 - \exp\left(-\left(\frac{x}{u}\right)^2\right)$ $x > 0$	$\mu = \frac{u}{2}\sqrt{\pi}$ $\sigma^2 = u^2(1-\pi/4)$ $V = 52.3\%$
Gamma (Erlang, when *k* = any positive integer)	$f_X(x) = \lambda \frac{(\lambda x)^{k-1}}{\Gamma(k)} e^{-\lambda x}, \quad x > 0$ where $\Gamma(k) = \text{gamma fn.} = \int_0^\infty t^{k-1} e^{-t} dt$ k any positive real number	$F_X(x) = \frac{\Gamma(\lambda x, k)}{\Gamma(k)}$ where $\Gamma(x, \alpha) =$ incomplete gamma fn $= \int_0^x e^{-t} t^{\alpha-1} dt$	$\mu_X = k/\lambda$ $\sigma_X^2 = k/\lambda^2$
Normal (Standard normal when μ = 0, σ = 1)	$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right),$ $-\infty < x < \infty$	Not available in closed form. Can be given in terms of the standard normal CDF, Φ: $F_X(x) = \Phi\left(\frac{x-\mu_X}{\sigma_X}\right)$	Obvious
Lognormal (exponentiated normal)	$f_X(x) = \frac{1}{\sqrt{2\pi}\zeta x} \exp\left[-\frac{1}{2}\left(\frac{\ln x - \lambda}{\zeta}\right)^2\right], \quad x > 0$	Not available in closed form. Can be given in terms of the standard normal CDF, Φ: $F_X(x) = \Phi\left(\frac{\ln x - \lambda}{\zeta}\right)$	$\zeta = \sigma_{\ln X} = \sqrt{\ln(1+V_X^2)}$ $\lambda = \mu_{\ln X} = \ln(\mu_X) - \frac{1}{2}\zeta^2$

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Chi-squared with *n* dof (sum of *n* independent squared standard normal variables) (Gamma with $k = n/2$ and $\lambda=1/2$)	$f_X(x) = \frac{1}{2^{n/2}\Gamma(n/2)} e^{-x/2} x^{n/2-1}, x \geq 0$ *n* does not have to be integer	$F_X(x) = \frac{\Gamma(x/2, n/2)}{\Gamma(n/2)}, \quad x > 0$	$\mu_X = n$ $\sigma_X^2 = 2n$
Chi with *n* dof (square root of Chi-squared random variable with dof *n*) Chi with $n = 1$ is called "half normal", with $n = 2$ is Rayleigh, and $n=3$ is MB	$f_X(x) = \frac{1}{2^{n/2-1}\Gamma(n/2)} e^{-x^2/2} x^{n-1}, x \geq 0$	$F_X(x) = \frac{\Gamma(x^2/2, n/2)}{\Gamma(n/2)}, \quad x > 0$	$\mu_X = \sqrt{2}\frac{\Gamma\left((n+1)/2\right)}{\Gamma\left(n/2\right)}$ $\sigma_X^2 = n - \mu_X^2$
Student's t distribution (ratio of standard normal to chi with dof *n*)	$\frac{\Gamma\left((n+1)/2\right)}{\sqrt{\pi n}\Gamma(n/2)}(1 + x^2/n)^{-(n+1)/2}, -\infty < x < \infty$		$\mu = 0$ $\sigma^2 = \frac{n}{n-2}, n > 2$
F distribution (ratio of two chi-squared random variables with dofs *m* and *n*)	$\frac{\Gamma\left((m+n)/2\right)}{\Gamma\left(m/2\right)\Gamma\left(n/2\right)}\left(\frac{m}{n}\right)^{m/2} x^{m/2-1}\left(1+\frac{mx}{n}\right)^{-(m+n)/2}, x > 0$		$\mu = \frac{n}{n-2}, n > 2$ $\sigma^2 = \frac{n^2(2m+2n-4)}{m(n-2)^2(n-4)}, n > 4$
Maxwell Boltzmann (Chi with n=3)	$f_X(x) = \frac{\sqrt{2}}{\sqrt{\pi}} e^{-x^2/2} x^2, x \geq 0$	$F_X(x) = \frac{\Gamma(x^2/2, 3/2)}{\sqrt{\pi}/2}, \quad x > 0$	$\mu_X = \sqrt{2}\frac{2}{\sqrt{\pi}}$ $\sigma_X^2 = 3 - \frac{8}{\pi}$
Gumbel (for maxima)	$f_X(x) = \alpha e^{-\alpha(x-u)} e^{-e^{-\alpha(x-u)}},$ $-\infty < x < \infty$	$F_X(x) = e^{-e^{-\alpha(x-u)}},$ $-\infty < x < \infty$	$\sigma_X = \frac{\pi}{\sqrt{6}\alpha}$ $\mu_X = u + \frac{0.5772}{\alpha}$
Frechet (for maxima)	$f_X(x) = \alpha k (x-\lambda)^{-k-1} e^{-\alpha(x-\lambda)^{-k}}$ $k, \alpha > 0, x > \lambda$	$F_X(x) = e^{-\alpha(x-\lambda)^{-k}}$ $k, \alpha > 0, x > \lambda$	$\mu = \lambda + \alpha^k \Gamma(1 - 1/k)$ $\sigma^2 = \alpha^{2k}\begin{bmatrix}\Gamma(1-2/k) - \\ \Gamma^2(1-1/k)\end{bmatrix}$

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Weibull (for minima) (Two parameter Weibull with x_0 = 0 is most common)	$f_X(x) = \frac{k}{u}\left(\frac{x-x_0}{u}\right)^{k-1}\exp\left(-\left(\frac{x-x_0}{u}\right)^k\right),$ $x > x_0$	$F_X(x) = 1-\exp\left(-\left(\frac{x-x_0}{u}\right)^k\right)$ $x > x_0$	$\mu = x_0 + u\,\Gamma(1+1/k)$ $\sigma^2 = u^2\left[\begin{matrix}\Gamma(1+2/k)\\ -\Gamma^2(1+1/k)\end{matrix}\right]$
Wald (inverse Gaussian) (time taken by a Brownian particle to reach distance *d* for the first time with drift velocity *v* and diffusion coefficient β. Here $\mu = d/v, \lambda = d^2/\beta$)	$f_X(x) = (\frac{\lambda}{2\pi x^3})^{\frac{1}{2}}\exp\left[-\frac{\lambda}{2\mu^2}\frac{(x-\mu)^2}{x}\right], x > 0$		$\mu_X = \mu$ $\sigma_X{}^2 = \mu^3/\lambda$
Cauchy	$f_X(x) = \frac{1}{\pi\lambda}\left[1+\left(\frac{x-\theta}{\lambda}\right)^2\right]^{-1}, \lambda > 0$ $\theta =$ location parameter, $\lambda =$ scale parameter	$F_X(x) = \frac{1}{2}+\frac{1}{\pi}\tan^{-1}\frac{x-\theta}{\lambda}$	μ_X does not exist. No finite moment of order 1 or greater exists
Beta	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1},$ $0 < x < 1, \alpha > 0, \beta > 0$		$\mu = \frac{\alpha}{\alpha+\beta}$ $\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
Pareto (“heavy tailed”)	$f_X(x) = kk_0x^{-k}$ $x > x_0$	$F_X(x) = 1-k_0x^{-k}$ $k_0, k > 0,$ $x > x_0 = \frac{1}{k_0^k} > 0$	

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