CHAPTER 18. COMPONENT RELIABILITY: CAPACITY – DEMAND TYPE

In this chapter we consider various definitions of failure, limit states, basic variables and computation of failure probability for "capacity demand" type compoents. Capacity demand type formulation occurs naturally where physics based approach to failure is needed. It is common for mechanical and structural components. We focus on components in this chapter. System reliability is taken up in CHAPTER 19.

18.1 Time to failure

Recall that the definition of failure given in Eq. (16.1) simplifies to Eq. (17.2)) for component reliabilty formulation where only one critical location and one failure mode of the structure is considered.

Refer to the definition of the safe set in Eq (17.2). For a component with time dependent capacity C(τ) and load Q(τ), the "safety margin" is described as the stochastic process:

$$M(\tau) = C(\tau) - Q(\tau), \quad 0 \le \tau \le t \tag{18.1}$$

In general the safety margin can be a more involved function of several time dependent quantities. The random time to failure is the instant that the safety margin exceeds a suitably defined safe set, $\mathbb{S}$, for the first time:

$$\text{First passage time: } T_f = \inf[t : M(t) \notin \mathbb{S},\ t > 0] \tag{18.2}$$

The first passage time (also called the first excursion time) defined this way is identical to the time to failure (TTF) discussed in CHAPTER 17.

18.2 Formulation of capacity-demand reliablity problems

In capacity demand type reliability, two broad classes of problems are found to occur.

a) *Cumulative damage/ fatigue failure.* Here the safety margin is a monotonic function of time. The safety margin is $M(t) = D_a - D(t)$ where D_a is the maximum allowable damage, and $D(t)$ is the cumulative damage. No healing is considered. Since the safety margin is a monotonic (decreasing) function of time, the time to failure is simply the point where the cumulative damge equals the critical value: $T_f = D^{-1}(D_a)$.

b) *Overload failure.* Here the safety margin is not a monotonic function of time. The failure event is defined as:

$$\{\text{Failure}\} = \{C(t) \ < \ Q(t),\ \text{for any } 0 < \tau \le t\} \tag{18.3}$$

$$R(t) = P[C(\tau) - Q(\tau) > 0 \text{ for all } \tau \in (0,t]]$$
$$\text{where: } Q(\tau) = D + L(\tau) + W(\tau) + S(\tau) + ... \tag{18.4}$$

We look at the first passage problem in increasing levels of complexity. We start with the case when the variables are time-invariant.

18.3 Case 1: Both C and Q are time invariant

$R(\tau)=R_0$ (no time dependence), $Q(\tau)=Q_0$ (sustained load)

$C(\tau) = C_0$ (no aging)

$Q(\tau) = Q_0$ (sustained load)

$$\mathrm{R}(t) = P[C_0 - Q_0 > 0]$$

This boils down to the random variable based treatment of reliability. The function has no explicit time dependence.

The reliability can be given in terms of the joint PDF of *C* and *Q*:

$$R(t) = \int_{-\infty}^{\infty}\int_{c=q}^{\infty} f_{C_0,Q_0}(c,q)\,dc\,dq \tag{18.5}$$

If C and Q are independent, the reliability function simplifies to:

$$\begin{aligned} R(t) &= \int_{-\infty}^{\infty}\int_{c=q}^{\infty} f_{C_0}(c) f_{Q_0}(q)\,dc\,dq = \int_{-\infty}^{\infty}(1 - F_{C_0}(q)) f_{Q_0}(q)\,dq \\ &= \int_{-\infty}^{\infty} F_{Q_0}(c) f_{C_0}(c)\,dc \end{aligned} \tag{18.6}$$

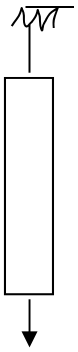
18.3.1.1 <u>Example: proof loading</u>

A structure has exponentially distributed capacity with mean μ_C. The load, independent of the capacity, also is exponential with mean μ_D.

a. Find the reliability of the structure.

b. A proof load test is performed on the structure as follows. A known load, c_0, is placed on the structure, and the structure survives without any damage. With this new information, find the updated reliability of the structure.

18.3.1.2 <u>Example: A small structural design problem</u>

	Consider a cable in a suspension bridge made of A36 steel with random yield strength Y (time invariant). It is a one RV problem. The axial load q = 1600 kip and the cross sectional area a = 50.3 in^2 are deterministic. Let cable failure be defined as yield of the gross section. Find the failure probability of the cable. The target failure probability is 0.001. Redesign if necessary.

Let Y be Weibull distributed with COV is 15%. The mean yield strength of A36 steel is 38 ksi. The shape and location parameters of Y are therefore, $V_Y = 15\% \Rightarrow k = 8$ and , $u = \dfrac{\mu}{\Gamma(1+1/8)} = \dfrac{38}{.94} = 40.4 \text{ ksi}$. The failure event is:

$$\{\text{Failure}\} = \left\{ \frac{q}{a} > Y \right\} \tag{18.7}$$

The probability of failure,

$$\begin{aligned} P_f &= P[\text{failure}] = P\left[Y < \frac{1600\,\text{kip}}{50.3\,\text{in}^2} \right] \\ &= P[Y < 31.8] = 1 - e^{-\left(\frac{31.8}{40.4}\right)^8} = 0.14 \end{aligned} \tag{18.8}$$

is solved using the Weibull CDF.

Since it is required that $P[\text{failure}] < .001$, the cable is inadequate. Reliability can be increased in four ways for this problem: increasing the area, reducing the load, increasing the mean strength, and decreasing the variability of strength. Of these, the second is not possible without restricting traffic, and the third and fourth would require a different material and possibly be very expensive. Thus, we decide to first try to increase the cross-sectional area.

The revised cross-sectional area can be found by finding the inverse of the CDF at the target P_f:

$$\therefore P\left[Y < \frac{q}{a_{new}} \right] = .001 \Rightarrow 1 - e^{-\left(\frac{q}{a_{new}\,40.4}\right)^8} = .001. \tag{18.9}$$

which yields,

$$a_{new} = \frac{1600}{40.4 \times .4217} = 93.9\,\text{in}^2 \tag{18.10}$$

Suppose the resultant diameter, about 11 inches, proves to be impractical. The next option is to try a different grade of steel without changing the diameter. Assume the distribution of Y_{new} remains Weibull and its COV remains 15%. The approach now is to select a new mean. The target probability of failure remains 0.001:

$$P\left[Y_{new} < \frac{q}{a}\right] = .001 \tag{18.11}$$

which yields:

$$\begin{aligned} &\exp\left[-\left(\frac{31.8}{u_{new}}\right)^8\right] = .999 \\ &\Rightarrow u_{new} = 75.4 \\ &\Rightarrow \mu_{new} = 75.4\Gamma\left(1+1/8\right) = 70.9 \text{ ksi} \end{aligned} \tag{18.12}$$

The new mean strength is acceptable provided this new grade of steel has sufficient ductility, corrosion resistance and other desirable properties. Otherwise, a totally new design may need to be adopted.

18.3.1.3 Example: A power distribution problem involving two variables:

On a certain day the power supply system for a large city has a capacity, *C*, which is a Normal random variable with mean 8 GW and coefficient of variation (c.o.v.) 20%. Demand on the system arises from two sources: residential (*R*) and Industrial (*I*). The residential power demand, R, is a Normal variable with mean 2 GW and c.o.v. 20%. The industrial power demand, *I*, is also a Normal variable with mean 3 GW and c.o.v. 30%. There is a slight dependence between the two demands: the correlation coefficient between *R* and *I* is 0.2. The system loses some power in transmission: the loss is constant and equal to 0.2 GW. For simplicity assume that *C*, *R* and *I* are time-invariant, and that the capacity of the power system is independent of the demands.

A "brownout" is said to occur if the total demand (plus transmission loss) exceeds *C*. Find the probability of a brownout in the city on the given day.

{Brownout} $=\left\{C < D\right\}$ where $D = R + I + 0.2$

$D = R + I + 0.2$

$\mu_D = \mu_R + \mu_I + a_0 = 2 + 3 + 0.2 = 5.2\ GW$

$\sigma_D^2 = \sigma_R^2 + \sigma_I^2 + 2\sigma_R\sigma_I\rho_{RI} = 0.4^2 + .9^2 + 2\times.4\times.9\times.2$

$\quad = 1.114$

$\sigma_D = 1.06$

D is Normal since it is a linear combination of Normal RVs. The performance margin is $Z = C - D$

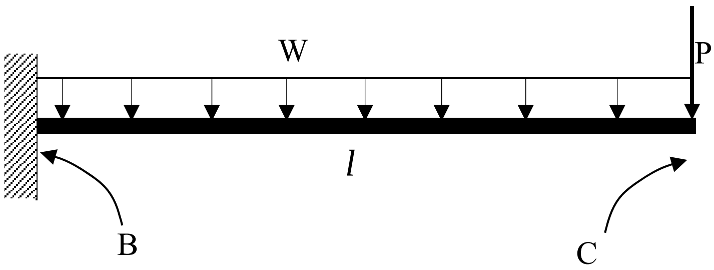
Z is normal since it is the difference of two normals. Its moments are:

$$\mu_Z = \mu_C - \mu_D = 8 - 5.2 = 2.8$$
$$\sigma_Z^2 = \sigma_C^2 + \sigma_D^2 = 1.6^2 + 1.114 = 3.67$$
$$\sigma_Z = 1.92$$

$$\text{P(Brownout)} = P[Z < 0] = \Phi\left[\frac{0-2.8}{1.92}\right] = \Phi(-1.46)$$
$$= 1 - .92785 = 0.07$$

18.3.1.4 <u>Example: A cantilevered beam</u>

A cantilevered beam is loaded by a uniformly distributed load, W, and a point load, P, as shown below. The beam length is 10 ft.



P and W are both normal random variables. Their means are 3 kip and 0.5 kip/ft, respectively, and their coefficients of variation (c.o.v.) are 15% and 20%, respectively. The correlation coefficient between them is $\rho_{PW} = 0.2$.

The yield strength, Y, is deterministic and equal to 36ksi. The section modulus, S, is random and independent of the loads. S is known to have a normal distribution with a c.o.v. of 7%.

Consider bending failure only. Choose the mean section modulus such that the beam has a reliability of 0.999.

18.4 Case 2: either C or Q or both vary non-randomly in time

At a given location and for a given failure mode, let the capacity and demand vary deterministically in time:

$$C(\tau) = C_0 \; d(\tau)$$
$$Q(\tau) = Q_0 \; h(\tau) \qquad (18.13)$$

C_0 and D_0 are random variables, and d, h are non-random functions of time, $d > 0, h > 0$. That is, if the process $C(\tau)$ is known at any instant t_1, its value can be known precisely at all other instants of time; likewise for $Q(\tau)$. Due to the non-random nature of d and h, the reliability function,

$$R(t) = P[C_0 \; d(\tau) - Q_0 \; h(\tau) > 0, \text{ for all } \tau \in (0,t]] \qquad (18.14)$$

can be written as:

$$R(t) = P[C_0 - Q_0 \max_{0<\tau\le t} \frac{h(\tau)}{d(\tau)} > 0] \tag{18.15}$$

In particular, d is the "aging" function. Its form can be derived from the mechanics of damage growth (e.g., corrosion loss[74], fatigue crack growth etc.) and the loading history. d=1 implies the capacity does not degrade with time, and h=1 implies the load is sustained in time. The above approach will be still valid for several simultaneously occurring loads (cf. Eq.(18.4)) if :

$$Q_0 \; h(\tau) = Q_0^{(1)} \cdot h_1(\tau) + Q_0^{(2)} \cdot h_2(\tau) + Q_0^{(3)} \cdot h_3(\tau) + \ldots \tag{18.16}$$

in which the h's are non-random functions of time and the individual loads $Q_0^{(i)}$ are random variables.

18.4.1 Monotonically decreasing strength and time-invariant load

This special case simplifies to:

$$R(t) = P[C_0 d(t) - Q_0 > 0], \; dd(\tau)/d\tau \le 0 \tag{18.17}$$

where d(t) is the monotonically decreasing aging function.

18.5 Case 3: Load occurs as a pulsed sequence with random magnitudes

18.5.1 Known number of load pulses and no aging

We first consider the case when C is time invariant (i.e., $d \equiv 1$ in Eq.(18.13)) but the load occurs as pulses of random magnitude $Q_1, Q_2, \ldots, Q_{n(t)}$ with the number of load pulses n in time t being known. We assume that the loads are IID, that is Q_i's are mutually independent and each Q_i has the same distribution F_Q. Further, the loads are indepedent of the capacity. The reliability function,

$$R(t) = P[Q_1 < C_0, Q_2 < C_0, Q_3 < C_0, \ldots, Q_{n(t)} < C_0] \tag{18.18}$$

can be simplified by first conditioning it on an aribtrary value of C_0, and using the IID property of the Q_i's:

$$R(t \mid C_0 = c) = [F_Q(c)]^{n(t)} \tag{18.19}$$

The total probability theorem is then applied to yield:

$$R(t) = \int_0^\infty [F_Q(c)]^{n(t)} f_{C_0}(c)\, dc \tag{18.20}$$

18.5.2 Q is a Poisson pulse process and no aging

We generalize the above situation of a known number of IID loads (and independent of capacity) and consider the loads to occur according to a Poisson pulse process (with rate λ). The magnitude of the pulses are IID as before. No aging is considered as before. Since the number of pulses in time interval (0,t] is random, the

$$\begin{aligned} R(t) &= \sum_{n=0}^{\infty} P\left[\bigcap_{i=1}^{n} Q_i < C_0 \middle| N(t)=n\right] P[N(t)=n] \\ &= \int_{c=0}^{\infty} \sum_{n=0}^{\infty} P\left[\bigcap_{i=1}^{n} Q_i < c \middle| N(t)=n, C_0=c\right] P[N(t)=n] f_{C_0}(c)dc \end{aligned} \tag{18.21}$$

By using the form of the Poisson PMF, the reliability function simplifies to:

$$R(t) = \int_0^{\infty} e^{-\lambda t(1-F_Q(c))} f_{C_0}(c)\, dc \tag{18.22}$$

18.5.3 Q is a Poisson pulse process and strength changes deterministically

We now introduce aging, as in Eq. (18.13). Since the loads occur as a Poisson pulse, the occurrence times, T_i, are random in nature, and the individual limit states are evaluated at these random instants of time:

$$R(t) = \sum_{n=0}^{\infty} P\left[\bigcap_{i=1}^{n} Q_i < C_0\, d(T_i) \middle| N(t)=n\right] P[N(t)=n] \tag{18.23}$$

Since these random occurrence times are ordered, $T_1 < T_2 < \ldots < T_i < T_{i+1} < \ldots$, their conditional joint PDF given that n pulses occured in (0,t] is $1/t^n$ (cf. Eq (12.6)). The reliability function, conditioned on a fixed value of C_0, then can be written as:

$$\begin{aligned} R(t \mid C_0 = c) &= \sum_{n=0}^{\infty} \iiint_{\text{all}\,\tau_i} P\left[\bigcap_{i=1}^{n} Q_i < c\, d(\tau_i) \middle| N(t)=n, T_i=\tau_i, T_i<T_j, 1\le i<j\le n\right] \times \\ &\qquad f_{\underline{T}}(\underline{\tau}) d\underline{\tau} P[N(t)=n] \quad (18.24) \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{t}\int_{\tau=0}^{t} F_Q\left[c\, d(\tau)\right] d\tau\right]^n P[N(t)=n] \end{aligned}$$

By using the form of the Poisson PMF, and removing the conditioning on C_0, the reliability function simplifies to:

$$R(t) = \int_0^{\infty} e^{-\lambda t\left(1-\frac{1}{t}\int_{\tau=0}^{t} F_Q[c\, d(\tau)]d\tau\right)} f_{C_0}(c)\, dc \tag{18.25}$$

Note that Eq (18.25) reduces to Eq (18.22) when *d* is identically equal to 1.

18.6 Case 4: both C and Q are random and time variant, but stationary

We now come to the more general case when $d(\tau)$ in Eq (18.13) is a stochastic process. The rate at which the margin process $M(\tau) = C(\tau) - Q(\tau)$ crosses the zero barrier (i.e., enters or leaves the "safe" domain) at an arbitrary time *t* is given by the joint PDF of the process and its derivative, $\dot{M}$:

$$\bar{\nu}_0(t) = \int_{-\infty}^{\infty} |\dot{m}(t)|\, f_{M(t)\dot{M}(t)}(0, \dot{m})\, d\dot{m} \tag{18.26}$$

If the margin process is stationary, the passages into the unsafe domain becomes asymptotically Poisson, so that the reliability function represents the probability of the first passage into the unsafe domain beyond time *t*:

$$\mathrm{R}(t) = \left(1 - F_T(0)\right) e^{-\nu_0^- t} \tag{18.27}$$

$F_T(0)$ is the probability that the margin is negative at $t = 0$ and is assumed to be small. In this case, the constant rate of downcrossing (into the unsafe domain) is:

$$\nu_0^- = \int_0^{\infty} \dot{m} f_{M\dot{M}}(0, \dot{m})\, d\dot{m} \tag{18.28}$$

Further, if the margin is stationary Gaussian, it is independent of its derivative at the same instant, and the downcrossing rate becomes:

$$\nu_0^- = \frac{\sigma_{\dot{M}}}{\sqrt{2\pi}} \frac{1}{\sigma_M} \phi\left(\frac{\mu_M}{\sigma_M}\right) \text{ if } M \text{ is stationary Gaussian} \tag{18.29}$$

The derivation is given below.