CHAPTER 11. POINT PROCESSES

11.1.1 Basic descriptors of a point process

Suppose these are m points at random locations $X_1, X_2, \ldots X_m$ in some region E of the Euclidean space R^d (e. g., location of diseased trees in an orchard, location of bugs in a computer code, arrival times of tracks on a bridge etc). Then the number of points in a set $A \subset E$ is :

$$N(A) = \sum_{k=1}^{m} I(X_k \in A) \tag{11.1}$$

where I is the indicator fn:

$$I(\bullet) = 1 \text{ if } \bullet \text{ is true}$$
$$= 0 \text{ otherwise}$$

This N is a point process on E with points $\{X_1....X_m\}$.

Requirements on a point process : bounded regions must always contain a finite number of points with probability 1, i. e. for any bounded set A, $P[N(A) < \infty] = 1$

Property of point process : For disjoint sets $B_1, B_2, \ldots$ in E : $N\left(\bigcup B_i\right) = \sum N(B_i)$

A Point Process is a random counting measure.

A stationary point process:

A point process N is stationary if the joint distribution of $N(B_1 + x)$, $N(B_1 + x)$, … $N(B_n + x)$ is independent of x for each $x \in E$, and B_1, B_2 in B (the class of bounded Borel sets in E)

A memoryless point process is a point process with independent increments.

A point process is simple if the occurrence times are disjoint almost surely.

Point process on $\mathbb{R}^+ = [0, \infty)$

When we talk about a point process on the real line, the points can simply be designated as “arrival times” (or “occurrence times” or “epochs”) : $t_1, t_2, \ldots, t_i \geq 0$.

The inter arrival times are $\tau_1 = t_1 - t_0 = t_1, \tau_2 = t_2 - t_1, \ldots.$

so that $t_n = \tau_1 + \tau_2 + \cdots + \tau_n$

(Kovalenko et at 1996)

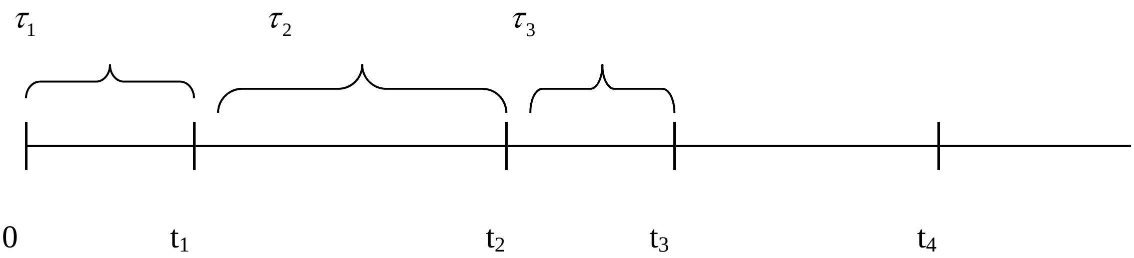
Point process can be equivalently defined by the joint distribution of :

1) The counting process N(t), or

2) The increments $N(t_i) - N(t_{i-1})$, or

3) The sequence of arrival times $\{t_n, n \geq 1\}$, or

4) The sequence of inter arrival times $\{\tau_n, n \geq 1\}$



If τ_i's are independent and identity distributed then the point process is a *renewal process* (also called recurrent point process , recurrent flow). Does a renewal process necessarily have independent increments, i.e., does a renewal process have to be memoryless?. No.

Let F_τ be the cdf of each τ_i i ≥ 2 and F_{τ_1} be the cdf of τ_1.

If $F_\tau \not\equiv F_{\tau 1}$, then the process is called “delayed”. Otherwise it is pure.

The renewal process is <u>ordinary</u> if $F_{\tau_1}(0) = 0, F_\tau(0) = 0$ i.e. no possibility of simultaneous occurrence of more than one events.

If $\overbrace{P[\tau_n < \infty]}^{F_\tau(\infty)} < 1$ then the renewal process consists of a finite random no. of points , and the process is called a generalized/ broken/ terminating / defective renewal process, and the occurrence time $t^* = \max\{t_n : t_n < \infty\}$ is called the *break point.*

The renewal function W is the mean of the counting process:

$$W(t) = E[N(t)] \tag{11.2}$$

and the renewal density is,

$$w(t) = \frac{dW}{dt} \tag{11.3}$$

such that the mean number of occurrences in time interval (a,b] is:

$$W(t) = E[N(a,b]] = \int_a^b w(t)dt \tag{11.4}$$

11.1.2 Poisson Process

Poisson process is a special renewal process. The “pure” or “uniform” or “homogeneous” Poisson model is a stationary renewal process with exponential inter-arrival times or, equivalently, a process with independent increments and a constant rate of occurrence. It is used to model a wide array of occurrence processes in various branches of science and engineering. In places where its assumptions are too restrictive, the pure Poisson process can be used as the building block for a large variety of processes showing clustering, dependence, non-stationarity etc.

Clustering phenomena can be accounted for by the Neymann-Scott and the Bartlett-Lewis processes (Cox and Isham 2000). In the former, the points of a pure Poisson process act as cluster centers so that a random number of cluster points are distributed independently and identically around each cluster center. In the Bartlett-Lewis process, on the other hand, the cluster points are generated according to a finite renewal process around the original Poisson process. A Polya process, which is a non-stationary version of the pure birth process, can also be used to model clustering (Wen 1990).

A point process is Poisson if:

1) No. of occurrences in a given time interval is independent of that in any other disjoint interval.
2) Probability of occurrence in a small interval Δt is proportional to Δt , i.e.
   $P[N(t,t+\Delta t)=1]=\lambda(t)\Delta t \qquad \Delta t \to 0$
3) Prob. of two or more occurrence in $t, t+\Delta t$ is 0 as $\Delta t \to 0$

The no. of occurrences, N_t, in the interval (0, t] is then a Poisson R.V. Its mean is the area under the rate curve:

$$m_t = \int_0^t \lambda(\tau)d\tau \tag{11.5}$$

Recall that a Levy process is a stochastic process with stationary independent increments. If the intervals are of equal length, the increments are iid. A Poission Process is thus a type of Levy process. It is also a pure birth process with a constant rate and thus comes under the general family of continous time Markov chains.

11.1.2.1 The homogeneous Poisson Process

A Poisson process with constant rate λ is called a homogeneous Poisson process.

Note "interval" is a general term: it does not necessarily signify time. It could equally well be distance along a line.

$$P_N(n,t) = P[N(0,t]=n] = P[N_t = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!} = \frac{e^{-\upsilon}\upsilon^n}{n!}, \upsilon = \text{mean.}$$

λ has units of 1/time.

For the random variable N_t, $\mu = \lambda t,\ \sigma^2 = \lambda t.$

The basic properties of the Poisson process are given below:

1) Given n points in $(0,t]$, t_i is $U(0,t)$

2) Distribution of $T_1, T_2, \ldots, T_n$ given $N(t) = n$ is

$$f_{T_1,T_2,\ldots,T_n|N(t)=n}(t_1,t_2\cdots t_n) = \begin{cases} \dfrac{n!}{t^n} & 0 < t_1 < \ldots < t_n < t \\ 0, & \text{otherwise} \end{cases} \tag{11.6}$$

Kingman p. 21: Every 1 – D Poisson Process can be transformed into one of constant rate, by means of a continuous monotonic transformation. This means in I-D only the uniform P.P. is of fundamental importance. and the properties of the most general locally finite process can be inferred from it.

11.1.2.2 Derivation of the Poisson Distribution

The Poisson Random Variable describes the no. of occurrences of a point process in a time interval occurring according to a Poisson experiment.

Proof :

$$\begin{aligned} P\left[N(0,t+dt]=n\right] &= P\left[N(0,t]=n \cap N(t,t+dt]=0\right]+P\left[N(0,t]=n-1 \cap N(t,t+dt]=1\right] \\ &= P\left[N(0,t]=n\right](1-\lambda dt)+P\left[N(0,t]=n-1\right]\lambda dt \end{aligned}$$

Define $p(n,t)=P\left[N(0,t]=n\right]$ and rewrite as:

$$p(n,t+dt)=p(n,t)(1-\lambda dt)+p(n-1,t)\lambda dt$$

Rearranging,

$$\frac{p(n,t+dt)-p(n,t)}{dt}=-\lambda p(n,t)+\lambda p(n-1,t)$$

That is,

$$\frac{dp(n,t)}{dt}=-\lambda p(n,t)+\lambda p(n-1,t)$$

or $\dfrac{dp(n,t)}{dt}+\lambda p(n,t)=\lambda p(n-1,t)$

Multiply both sides by the integrating factor $e^{\lambda t}$ and write:

$$d\left(p(n,t)e^{\lambda t}\right)=\lambda p(n-1,t)e^{\lambda t}$$

The solution is :

$$p(n,t)e^{\lambda t}=\lambda\int p(n-1,t)e^{\lambda t}dt+c_n$$

First set n = 0, and obtain:

$$p(0,t)e^{\lambda t}=\lambda\int \underbrace{p(-1,t)}_{=0}e^{\lambda t}dt+c_0$$

$$\therefore\ p(0,t)=c_0e^{-\lambda t}$$

Since there can be no occurrence before counting starts, we must have:

$$\begin{aligned} &p(0,0)=1 \\ &p(n,0)=0 \text{ for } n\geq 1 \end{aligned} \tag{11.7}$$

Thus, $1 = c_0 e^{-0}$

$\therefore c_0 = 1.$

$\therefore p(0,t) = e^{-\lambda t}$

Now set n = 1:

$$\therefore p(1,t)e^{\lambda t} = \lambda \int p(0,t)e^{\lambda t} dt + c_1 = \lambda \int e^{-\lambda t} e^{\lambda t} dt + c_1$$
$$= \lambda t + c_1$$

Since $p(1,0) = 0$, we have $c_1 = 0$

Thus, $p(1,t) = e^{-\lambda t} \lambda t$

Set n=2:

$$\therefore p(2,t)e^{\lambda t} = \lambda \int p(1,t)e^{\lambda t} dt + c_2 = \lambda \int e^{-\lambda t} \lambda t e^{\lambda t} dt + c_2$$
$$= (\lambda t)^2 / 2 + c_2$$

Since $p(2,0) = 0$, we have $c_2 = 0$, giving us:

$p(2,t) = e^{-\lambda t} \dfrac{(\lambda t)^2}{2!}$ etc.

Use induction to come up with:

$$p(n,t) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}$$

11.1.2.3 Erlang Distribution

Let T_k = time to the k^{th} arrival (in a Poisson process)

$P[T_k \le t] = P[N(t) \ge k]$

$$F_{T_k}(t) = \sum_{n=k}^{\infty} \frac{e^{-\lambda t} (\lambda t)^n}{n!}$$

Or

$$\therefore f_{T_k}(t) = \sum_{n=k}^{\infty} \left[-\lambda e^{-\lambda t} \frac{(\lambda t)^n}{n!} + e^{-\lambda t} \frac{n(\lambda t)^{n-1} \lambda}{n!} \right]$$

$$= \lambda e^{-\lambda t} \sum_{n=k}^{\infty} \left[\frac{n(\lambda t)^{n-1} - (\lambda t)^n}{n!} \right]$$

$$= \lambda e^{-\lambda t} \left[\sum_{n=k}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} - \sum_{n=k}^{\infty} \frac{(\lambda t)^n}{n!} \right]$$

$$= \lambda e^{-\lambda t} \left[\sum_{n=k-1}^{\infty} \frac{(\lambda t)^n}{n!} - \sum_{n=k}^{\infty} \frac{(\lambda t)^n}{n!} \right]$$

$$= \lambda e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!}$$

Mean of Erlang distribution:

$$T_k = \tau_1 + \tau_2 + \cdots + \tau_k$$
$$\mu_{T_k} = 1/\lambda + 1/\lambda + ... + 1/\lambda = k/\lambda$$

Variance of Erlang distribution:

$$\sigma_{Tk}^{\ 2} = \text{var}(\tau_1) + \text{var}(\tau_2) + \cdots$$
$$= \frac{1}{\lambda^2} + \cdots + \frac{1}{\lambda^2} = \frac{k}{\lambda^2}$$

Erlang generalizes to gamma when *k* is not an integer.

11.1.2.4 Non-homogeneous Poisson processes

If the rate λ is not constant (but varies deterministically with time), the Poisson process is termed as non-homogeneous. The number of occurrences, N_t, in time (0,t] is still Poisson distributed:

$$P[N_t = n] = \frac{e^{-m(t)}(m(t))^n}{n!} \tag{11.8}$$

with mean

$$m(t) = \int_0^t \lambda(\tau) d\tau \tag{11.9}$$

Every non-homogeneous Poisson Process can be transformed into one of constant rate, by means of a continuous monotonic transformation and the properties of the locally finite process can be inferred from it(Kingman 2002).

The random occurrence times of a homogeneous Poisson process can be transformed in into an appropriate non-homogenoues PP as follows (Bedford and Cooke p. 57):

Let T_1, T_2, T_3, … be the random occurrence times of a homogeneous PP with rate unity.

Define:

$$\phi(x) = \inf\{t \mid m(t) \geq x\} \tag{11.10}$$

where $m(t)$ is given by Eq (11.9). Then the sequence $\phi(T_1), \phi(T_2), \phi(T_3), ...$ is the occurrence times of the non-homogeneous process with rate $\lambda(t)$.

11.1.2.5 Mixed Poisson Process

The rate is now a random variable, but is not a function of time.

11.1.2.6 Compound Poisson process

A process with random jumps in the state occuring at random points in time. The random points in time occur according to a Poisson process. At each random point, the state undergoes a jump of random magnitude according to a specified distribution.

11.1.2.7 Spatial Poisson processes

11.1.2.8 Simulation of Poisson Processes

11.1.2.8.1 Homogeneous Poission processes

Let the homogeneous PP have constant rate λ.

1. Generate IID exponentials with rate λ, these are the interarrival times. Hence get the occurrence times. If the life time is *t*, then continue until *t* is exceeded.

2. Given length of time *t*, generate a Poisson random variable *N* with mean λ*t*. Let the realization of *N* be *n*. Generate *n* independent uniforms between 0 and *t*. These are the random occurrence times.

11.1.2.8.2 Non-homogeneous Poisson processes

If the rate is bounded, let

$$\lambda(\tau) < \lambda_m, \qquad 0 < \tau \le t \tag{11.11}$$

Then a homogeneous Poisson process, $X(t)$, with rate λ_m may be thinned to obtain the desired non-homogeneous Poisson process, $Y(t)$, with rate $\lambda(t)$ as follows.

Generate points $\{x_i(t_i)\}$ from $X(t)$. Also generate an IID U(0,1) sequence $\{u_i\}$. Retain the point $x_i(t_i)$ if $u_i \le \lambda(t_i)/\lambda_m$ and thus populate $\{\mathrm{y}_i(t_i)\}$. Proof is given in Ross.

11.1.2.8.3 Spatial Poisson Processes

11.1.3 Cox process

A more versatile generalization of the pure Poisson process occurs if the rate, $\Lambda(t) \ge 0$, itself is considered to be a random process yielding what is known as a doubly stochastic Poisson process (or Cox process) (Cox and Isham 2000). The mean measure of the point process in the interval [0,*t*] is a random variable and is given by:

$$M_t = \int_0^t \Lambda(\tau) d\tau \tag{11.12}$$

Then, conditioned on $M_t = m_t$ (where m_t is any positive number for given *t*), the point process *N*(*t*) becomes a (generally non-homogeneous) Poisson process, i.e., the counts are distributed according to:

$$P\left[N(t) = x \mid M_t = m_t\right] = e^{-m_t} \frac{(m_t)^x}{x!} \tag{11.13}$$

Special cases of the Cox process include the following: $\Lambda(t) = \lambda$ (a constant) yield the *homogeneous* Poisson process; if $\Lambda(t) = \lambda(t)$ is a non-random function of time, we get a *non-homogeneous* Poisson process; and if $\Lambda(t) = \Lambda$ is a time-independent random variable, we are left with a *mixed* Poisson process.

μ is a random measure on E . Let a counting process N(A) be defined on E. If given $\mu(A)$, N(A) is Poisson, then N(A) is a Cox Process.

Var [N(A)] ≥ E[N(A)]. The two are equal when μ is nonrandom.

11.1.4 Polya process

$$P[N_t = n] = \frac{(\lambda t)^n}{n!}(1+\beta\lambda t)^{-(n+1/\beta)}\prod_{i=1}^{n-1}(1+\beta i) \tag{11.14}$$

In the limit β=0, it boils down to the Poisson process.