EE60039 – Autumn Semester 2023-24

Exercise 6: Probability and Random Processes for Signals and Systems

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Q 6.1: Properties of Covariance

Let X, Y, Z be random vectors. Show the following.

- $\mathbb{E}[(AX)(CY)^\top] = A\mathbb{E}[XY^\top]C^\top$.
- $\texttt{cov}(AX + b, CY + d) = A\texttt{cov}(X, Y)C^\top$.
- $\texttt{cov}(AX + b,) = A\texttt{cov}(X)A^\top$.
- $\texttt{cov}(X + Y, Z) = \texttt{cov}(Y, Z) + \texttt{cov}(X, Z)$.

Q 6.2: Properties of Covariance

Let X be a random vector. Show that $\texttt{cov}(X)$ is a positive semi-definite matrix.

Q 6.3: Scalar State Estimation

Consider the discrete-time scalar dynamical system:

$$x_{k+1} = ax_k + w_k,$$
$$y_k = cx_k + v_k,$$

where $a, c \in \mathbb{R}$ and w_k, v_k are zero-mean scalar random variables with variances σ_w^2 and σ_v^2 satisfying the standard assumptions that we used to derive Kalman filter equations. Let x_0 be deterministic and known, i.e., $\texttt{VaR}(x_0) = 0$. Find the Kalman gain L_1 and L_2 and $\widehat{x}_{1|1}$ and $\widehat{x}_{2|2}$. What are the values of the above quantities when $\sigma_w = 0$? Explain intuitively why you obtain these results. Repeat the above question when $\sigma_v = 0$ (this time $\sigma_w \neq 0$) and explain intuitively why you obtain such results.

Q 6.4: Steady-State Kalman Gain

Find the steady-state Kalman gain and steady-state error covariance of the Kalman filter for the above system when $a = 1, c = 2, \texttt{VaR}(x_0) = \sigma_x^2 = 4, \sigma_w^2 = 0.5, \sigma_v^2 = 0.8$.

Q 6.5: State Estimation

Consider the discrete-time scalar dynamical system:

$$\begin{aligned} x_{k+1} &= 2x_k, \\ y_k &= x_k - x_{k-1} + v_k, \end{aligned}$$

where $\mathbb{E}[x_0] = \mathbb{E}[v_k] = 0$, v_k is uncorrelated in time and with x_0, $\sigma(x_0) = 1$ and $\sigma^2(v_k) = 0.25$. Find the LMSE estimate of x_1 given $y_1 = 1$. Can a Kalman filter be used to estimate the state x_k? If so, describe how.

Q 6.6: Position Estmation

In this problem, we will estimate the position and velocity of a particle on the 2D plain by implementing a Kalman Filter on Matlab. The system states at time k consists of the position on x and y axes and the velocity on x and y co-ordinates as

$$x(k) = \begin{bmatrix} p_x(k) \\ p_y(k) \\ v_x(k) \\ v_y(k) \end{bmatrix}.$$

The state update and measurement model are given by

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 0.9 & 0.4 \\ 0 & 0 & -0.4 & 0.9 \end{bmatrix} x(k) + \begin{bmatrix} T^2/2 & 0 \\ 0 & T^2/2 \\ T & 0 \\ 0 & T \end{bmatrix} w(k), \\ y(k) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(k) + v(k), \end{aligned}$$

where $w(k) \sim \mathcal{N}(0, Q)$ and $v(k) \sim \mathcal{N}(0, R)$ with $Q = I_2, R = 0.1 \times I_2$ and I_2 is the 2×2 identity matrix. Let $T = 0.1$. Let $\mathbb{E}(x_0) = \begin{bmatrix} 5 \\ 5 \\ 1 \\ 1 \end{bmatrix}$ and $\texttt{VaR}(x_0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$.

Generate a sample trajectory of the above system, and estimate the state from the outputs using a Kalman filter. Plot the estimated state vs. time. Plot the predicted output and given output vs. time on the same figure. Mark all axes and legends properly.

What are the eigenvalues of the matrix $A - LC$ where L is the steady-state Kalman gain? Find the square of the norm of the error between actual and predicted output (i.e., $\sum_{k=1}^{N} ||y(k) - \hat{y}_{k|k-1}||_2^2$).