EE60039 – Autumn Semester 2023-24

Exercise 5: Probability and Random Processes for Signals and Systems

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Q 5.1: Estimation Task

Let X be a random variable with Exponential distribution with parameter Λ, and Λ itself is a discrete random variable with $\mathbb{P}(\Lambda = 1) = \mathbb{P}(\Lambda = 2) = 0.5$. Find $\mathbb{E}[X]$. Suppose you observe that $X = 1$. Find the maximum likelihood and maximum a-posteriori estimates of Λ.

Recall: the density of an exponential random variable X with parameter Λ is $f_X(x) = \Lambda e^{-\Lambda x}, x \geq 0$.

Q 5.2: ML Estimation Task

Suppose $\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_N$ be N i.i.d. realizations of a random variable with density given by

$$f(x, \theta) = \begin{cases} \theta x^{\theta - 1}, & x \in (0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Find the maximum likelihood estimator of θ.

Q 5.3: Simple Estimation Task

Let X and Y be a random variables with joint pdf:

$$f_{X,Y}(x, y) = \begin{cases} x + y, & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

1. Find the MAP estimator of X given $Y = y$.
2. Find the ML estimator of X given $Y = y$.
3. Find the MMSE estimator of X given $Y = y$.
4. Find the LMSE estimator of X given $Y = y$.

Q 5.4: Another Simple Estimation Task

Let X be a random variable (r.v.) that denotes the temperature of a room. X is a discrete r.v. that can takes values in the finite set $\{30, 32, 34\}$ where the value corresponds to the temperature in degree Celsius. Let Y be a random variable that indicates the comfort level of an occupant. $Y = 1$ is the occupant is comfortable, $Y = -1$ otherwise. The following is known about the conditional probability of Y for different values of X.

$$\begin{aligned} p_{Y|X}(Y = 1|X = 30) &= 0.8, \quad p_{Y|X}(Y = -1|X = 30) = 1 - 0.8 = 0.2, \\ p_{Y|X}(Y = 1|X = 32) &= 0.6, \quad p_{Y|X}(Y = -1|X = 32) = 1 - 0.6 = 0.4, \\ p_{Y|X}(Y = 1|X = 34) &= 0.2, \quad p_{Y|X}(Y = -1|X = 34) = 1 - 0.8 = 0.8. \end{aligned}$$

1. Suppose an occupant feels comfortable ($Y = 1$). Then, what the maximum likelihood estimate of the room temperature X?

   Suppose X has a prior distribution (p.m.f.) given by $p_X(X = 30) = p_X(X = 32) = 0.5$, and $p_X(X = 34) = 0$. Then,

2. What is the posterior distribution of X if $Y = 1$?
3. What is the MAP estimate of X if $Y = 1$?
4. What is the MMSE estimate of X if $Y = 1$?
5. Repeat the above questions when $Y = -1$.

Q 5.5: Yet Another Simple Estimation Task

Consider a variant of the above problem where the level of comfort Y is a continuous r.v. with conditional distributions are given by

$$\begin{aligned} f_{Y|X}(y|X = 30) &\sim U[-1, 0], \\ f_{Y|X}(y|X = 32) &\sim U[-0.5, 0.5], \\ f_{Y|X}(y|X = 34) &\sim U[0, 1], \end{aligned}$$

where $U[a, b]$ denotes the uniform distribution between a and b.

1. Suppose we observe that $Y = 0.2$. Then, what the maximum likelihood estimate of the room temperature X?

   Suppose X has a prior distribution (p.m.f.) given by $p_X(X = 30) = p_X(X = 32) = 0.5$, and $p_X(X = 34) = 0$ as before. Then,

2. What is the posterior distribution of X if $Y = 0.2$?
3. What is the MAP estimate of X if $Y = 0.2$?
4. What is the MMSE estimate of X if $Y = 0.2$?
5. Repeat the above questions when the prior distribution of X is given by $p_X(X = 30) = p_X(X = 32) = 0.4$, and $p_X(X = 34) = 0.2$.

Q 5.6: Mean Square Estimation

Let $Y_i = X + V_i$, where X is a random variable we want to estimate, Y_i is the i-th observation, and V_i is a random variable which acts as measurement noise. Let X and V_i's be zero-mean and independent random variables with variance of X being $\sigma_X^2 = 1$ and variance of V_i be $\sigma_V^2 = 1$.

(a) Suppose we make two noisy observations Y_1 and Y_2. Find the linear mean-square estimator for X in terms of Y_1 and Y_2.

(b) (**Bonus**) Suppose we make N noisy measurements $Y_1, Y_2, \ldots, Y_N$. Find the linear mean-square estimator for X in terms of $Y_1, Y_2, \ldots, Y_N$. Use the fact that $(I + aa^{\mathsf{T}})^{-1} = I - \frac{1}{N+1}aa^{\mathsf{T}}$, where I is the identity matrix of dimension N and $a = [1 \quad 1 \quad \ldots \quad 1]^{\mathsf{T}}$ is the column vector of length N with all entires 1.

Q 5.7: Linear Mean Square Estimation

Assume that a zero-mean random variable $X = X_c + Z$ where Z is uncorrelated with the observation Y. Show that the LMSE of X given Y equals the LMSE of X_c given Y. In other words, $\widehat{\mathbb{E}}(X|Y) = \widehat{\mathbb{E}}(X_c|Y)$. This shows that LMSE does not give us any information about the component Z that is uncorrelated with the observation Y.

Q 5.8: Linear Mean Square Estimation

Let $Y = X + V$, where X and V are independent zero-mean Gaussian random variables with variances σ_X^2 and σ_V^2, respectively. Show that the LMSE estimator of X^2 given Y and Y^2 is

$$\hat{X}^2 = \sigma_X^2 + \frac{\sigma_X^4}{\sigma_X^4 + 2\sigma_X^2\sigma_V^2 + \sigma_V^4}(Y^2 - \sigma_X^2 - \sigma_V^2).$$

Hint: Use the fact that if Z is a zero-mean Gaussian r.v. with variance σ_Z^2, then $\mathbb{E}[Z^3] = 0$ and $\mathbb{E}[Z^4] = 3\sigma_Z^4$.

Q 5.9: Linear Mean Square Estimation

Let $Y_1 = (X + W_1)^2, Y_2 = W_2 X$. Find the LMSE estimator of X in terms of Y_1 and Y_2 (treat $Y = [Y_1 \quad Y_2]^{\mathsf{T}}$ as a random vector and use the appropriate formula for $\widehat{\mathbb{E}}[X|Y]$). Use the following information: $\mathbb{E}[W_1] = 0, \mathbb{E}[W_1^2] = 1, \mathbb{E}[W_2] = 2, \mathbb{E}[W_2^2] = 1, \mathbb{E}[X] = 1, \mathbb{E}[X^2] = 2, \mathbb{E}[X^3] = 3, \mathbb{E}[X^4] = 4$.

Q 5.10: Linear Mean Square Estimation

Let $y_k = v_k + v_{k-1}, k \geq 0$, where $v_j, j \geq -1$ is a zero-mean stationary white noise scalar process with unit variance. Show that

$$\hat{y}_{k+1|k} = \frac{k+1}{k+2}(y_k - \hat{y}_{k|k-1}),$$

where $\hat{y}_{k|m} = \widehat{\mathbb{E}}[y_k|Y_M]$ is the LMSE estimator of y_k given $Y_M = (y_1, y_2, \ldots, y_m)$.

Q 5.11: Prediction and Interpolation

Consider a series of observations $y(0), y(1), \ldots$ where

$$y(k) = a_0 w(k) + a_1 w(k-1) + \ldots + a_N w(k-N),$$

where $w(-N), w(-N+1), \ldots$ are independent with $\mathbb{E}[w(l)] = 0$ and $\mathbb{E}[w^2(l)] = \sigma_w$. The coefficients $a_0, \ldots, a_N$ are known. Solve the following problems.

1. Prediction: Find the LMSE predictor of $y(k+1)$ based on $y(k)$ (based only on $y(k)$, not $y(k-1)$, etc).
2. Interpolation: Find the LMSE predictor of $y(k)$ based (only) on $y(k-1)$ and $y(k+1)$ for $k > 1$.

Q 5.12: Mean Square Estimation

Let $s[n]$ be a discrete-time zero-mean wide sense stationary (WSS) signal with auto-covariance function given by

$$C_{ss}[m] = \begin{cases} 15, & \text{for} \quad m = 0, \\ -4, & \text{for} \quad m = 1, -1, \\ 5, & \text{for} \quad m = 2, -2, \\ 0, & \text{for} \quad m \geq 3. \end{cases}$$

Suppose we observe this signal with a time-delay and with some measurement noise $w[n]$. $w[n]$ is a zero-mean WSS signal uncorrelated with $s[n]$ and $C_{ww}[m] = 3$ if $m = 0$ and $C_{ww}[m] = 0$ for $m \neq 0$. The received signal is given by

$$r[n] = s[n-1] + w[n].$$

We would like to estimate $\hat{s}[n]$ that minimizes the mean square error (MSE) $\mathbb{E}\left[(s[n] - \hat{s}[n])^2\right]$.

1. Find the covariance functions $C_{rs}[m]$ and $C_{rr}[m]$ (in terms of $C_{ss}[m]$ and $C_{ww}[m]$).
2. If the estimator is to be of the form $\hat{s}[n] = ar[n]$, then find the value of a that minimizes the MSE.
3. If the estimator is to be of the form $\hat{s}[n] = a_0 r[n] + a_1 r[n-1]$, then find the values of a_0 and a_1 that minimizes the MSE.
4. If the estimator is to be of the form $\hat{s}[n] = \frac{1}{2} r[n - n_0]$ with the integer $n_0 \geq 0$, then find the value of n_0 that minimizes the MSE.

Q 5.13: Past End-Sem Exam Question

X is a three-dimensional random vector with $E[X] = 0$ and autocorrelation matrix R_X with elements $r_{ij} = (-0.80)^{|i-j|}$. Use X_1 and X_2 to form a linear estimate of $X_3 : \hat{X}_3 = a_1 X_1 + a_2 X_2$, i.e., determine a_1 and a_2 that minimizes mean-square error.