EE60039 – Autumn Semester 2023-24

Exercise 4: Probability and Random Processes for Signals and Systems

Prof. Ashish Ranjan Hota
Department of Electrical Engineering, IIT Kharagpur

Q 4.1: Markov Chain: Classification of States and Limiting Distribution

For the following transition probability matrices with state space $S \subseteq \{1,2,3,4,5,6\}$, determine

- all the communicating classes
- whether each communicating class is recurrent or transient
- the period of each communicating class
- whether a stationary distribution exists and is unique
- the stationary distribution when it is unique
- $\lim_{n\to\infty} P^n$ when it exists
- numerically simulate a trajectory of the Markov chain for initial state 1 and find the proportion of time it spends in each state from $n = 0$ to 50.
- numerically simulate 20 trajectories of the Markov chain for initial state 1 and find the proportion of time it is at each state when $n = 5$. Compare this with the first row of P^5.

1. $P_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

2. $P_2 = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \end{bmatrix}$. What do you notice about the transition probabilities and the graph structure?

3. $P_3 = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$ for some $a, b \in (0,1)$.

4. $P_4 = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$

5. $P_5 = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

6. $P_6 = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

Q 4.2: Markov Chain

Consider the integers $1, 2, 3, 4$ to be marked on the real line. Let X_n be a Markov chain that moves to the right with probability $\frac{2}{3}$ and to the left with probability $\frac{1}{3}$, subject to the rule that if X_n tries to go to the left from 1 or to the right from 4 it stays put. Find the transition probability matrix for the chain, and the limiting amount of time the chain spends at each integer.

Q 4.3: Markov Chain

Show that if P is the transition matrix of an irreducible chain with finitely many states, then $Q := \frac{1}{2}(I + P)$ is the transition matrix of an irreducible and aperiodic chain where I is the identity matrix. Show that P and Q have the same stationary distributions. Discuss, physically, how the two chains are related.

Q 4.4: Markov Chain

Consider i.i.d. coin tosses with probability of head being p. Let $X_i = 1$ if the i-th coin toss is head and $X_i = 0$ otherwise. Let $Y_0 = 0$ and $Y_n = \sum_{i=0}^{n-1} X_i$. Is $\{Y_n\}_{n=\{1,2,\ldots,N\}}$ a Markov Chain? Find the state space, transition probability matrix and the underlying graph when $N = 6$. Determine the recurrent and transient states.