EE60039 – Autumn Semester 2023-24

Exercise 3: Probability and Random Processes for Signals and Systems

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Q 3.1: Stationary Random Process

Let $X(t)$ be a continuous-time wide sense stationary (w.s.s.) random process with mean $\mathbb{E}[X(t)] = \mu_X$, and autocorrelation function $R_X(\tau)$. Then answer the following questions.

1. Let A be a nonnegative random variable independent of $X(t)$. Let $Y(t) := A + X(t)$. Show that $Y(t)$ is w.s.s.
2. Let $Z(t)$ be a w.s.s. random process independent of $X(t)$ with mean $\mathbb{E}[Z(t)] = \mu_Z$, and autocorrelation function $R_Z(\tau)$. Let $W(t) := X(t)Z(t)$. Find the mean and autocorrelation of $W(t)$ and show that it is w.s.s. Are $W(t)$ and $X(t)$ jointly w.s.s.?
3. Let $X_1(t) := X(t + a)$ where a is a constant. Is $X_1(t)$ w.s.s.? Are $X(t)$ and $X_1(t)$ jointly w.s.s.?
4. Let $X_2(t) := X(at)$. Is $X_2(t)$ w.s.s.? Are $X(t)$ and $X_2(t)$ jointly w.s.s.?
5. Let $X_3(t) := X(t - t_0)$ where t_0 is a constant delay. Is $X_3(t)$ w.s.s.? Are $X(t)$ and $X_3(t)$ jointly w.s.s.?

Q 3.2: Second Order Theory

Consider a discrete-time LTI system $y(k) = u(k) - 2u(k-1) + u(k-2)$. Let the input be a zero-mean wide sense stationary process with auto-correlation

$$R_{uu}[\tau] = \mathbb{E}[u(k)u(k-\tau)] = \begin{cases} 2, & \tau = 0, \\ 1, & \tau = 1, -1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the expression for the cross-correlation $R_{yu}[\tau]$.

(c) Completely determine $R_{yu}[\tau]$, i.e., for all values of τ.

(d) Completely determine $R_{yy}[\tau]$.

Q 3.3: Stationary Random Process

Consider a discrete-time LTI system $y(k) = u(k) + a_1 u(k-1)$. Let the input be a zero-mean wide sense stationary process with auto-correlation

$$R_{uu}[\tau] = \mathbb{E}[u(k)u(k-\tau)] = \begin{cases} 1, & \tau = 0, \\ 0.5, & \tau = 1, -1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the expression for the cross-correlation $R_{yu}[\tau]$.

(b) Suppose $R_{yu}[1] = 1$. Find the value of a_1.

(c) Completely determine $R_{yu}[\tau]$ (as has been done for $R_{uu}[\tau]$ above).

(d) Completely determine $R_{yy}[\tau]$.

Q 3.4: Stationary Process

Consider a discrete-time system $y(k) = u(k) + a_1 u(k-1)$. Let $\mathbb{E}[u(k)] = 0$ for all $k \geq 0$ and

$$\mathbb{E}[u(k)u(k-\tau)] = \begin{cases} 2, & \tau = 0, \\ 1, & \tau = 1, -1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Suppose $\mathbb{E}[y(k)u(k-1)] = 3$. Then determine the value of a_1.

(b) Does $\mathbb{E}[y(k)u(k-\tau)]$ depend on k? Determine $\mathbb{E}[y(k)u(k-\tau)]$ for $k = 2$ and $\tau = 0, 1, -1, 2$.

Q 3.5: Random Process and Stationarity

Consider a continuous-time random process $(X(t) : t \geq 0)$ with the following properties:

- $X(t) \in \{-1, 1\}$, that is, $X(t)$ takes value either -1 or 1. In other words, each $X(t)$ is a discrete random variable.
- $X(t)$ changes sign with a probabilistic process; the probability that there are k sign changes of $X(t)$ in an interval of length T has a Poisson distribution with parameter λT. In other words,

$$\mathbb{P}(\text{number of sign changes between } X(t+T) \text{ and } X(t) \text{ is } k) = \frac{(\lambda T)^k e^{-\lambda T}}{k!}.$$

- The p.m.f. of $X(0)$ is specified as $\mathbb{P}(X(0) = 1) = p$, and $\mathbb{P}(X(0) = -1) = 1 - p$, where $p \in (0, 1)$.

Then,

1. At $t = t_0$, what is the p.m.f. of the random variable $X(t_0)$?
2. Is the random process $X(t)$ wide sense stationary?

Q 3.6: Gaussian Process

Let $\{X_t\}_{t\in\mathbb{R}}$ be a stochastic process defined by $X_t = tA$ where $A \sim \mathcal{N}(0,1)$. Show that $\{X_t\}_{t\in\mathbb{R}}$ is a Gaussian Process. Find its mean and covariance.

Q 3.7: Gaussian Process

Let $\{W_n\}_{n\in\mathbb{N}}$ be a discrete-time stochastic process where each W_n has zero-mean, Gaussian, independent and identically distributed.

1. Let $\{S_n\}_{n\in\mathbb{N}}$ be another process defined as $S_k = \sum_{i=1}^{k} W_i$. Find the mean and covariance of the process $\{S_n\}_{n\in\mathbb{N}}$. Is this process W.S.S.?
2. Let $\{X_n\}_{n\in\mathbb{N}}$ be a discrete-time stochastic process where each $X_{n+1} = aX_n + W_n$ with $X_1 = 0$. Find the mean and covariance of the process $\{X_n\}_{n\in\mathbb{N}}$. Is this process W.S.S.?

Q 3.8: Independent and Stationary Increment Process

Let $\{X_t\}_{t\geq 0}$ be a stochastic process with independent and stationary increments and let X_0 be an arbitrary random variable. Show that $\mathbb{E}[X_t] = \mathbb{E}[X_0] + t\mathbb{E}[X_1 - X_0]$. Find $C_X(t_1, t_2)$ in terms of $\texttt{var}[X_0], \texttt{var}[X_1], t_1$ and t_2.