EE60039 – Autumn Semester 2023-24

Exercise 2: Probability and Random Processes for Signals and Systems

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Q 2.1: Function of a random variable

Let X be a random variable with C.D.F. denoted by F_X. Find the C.D.F. of the following:

1. $|X|$
2. $aX + b$
3. e^X

Q 2.2: Function of a random variable

Let X be a random variable uniformly distributed over $[-1, 1]$. Let $Y = \max(0, X)$. Find the C.D.F. of Y.

Q 2.3: Function of a uniform random variable

Let U be a random variable uniformly distributed over $[0, 1]$. We wish to construct a random variable from U which has C.D.F. F. Find a function g such that the C.D.F. of $g(U)$ is F.

Q 2.4: Minimum of Exponential Random Variables

Let $\{X_i\}_{i=1}^n$ be a collection of independent random variables with X_j having Exponential distribution with parameter λ_j. Let $a_i > 0, \quad i = 1, 2, \ldots, n$. Let $Y = \min_{i=1,2,\ldots,n}\{a_i X_i\}$. Prove that Y is Exponentially distributed and find $\mathbb{E}[Y]$.

Q 2.5: Characteristic Function

Let X be a random variable with characteristic function $\phi_X(h)$. If $Y = aX + b$, show that $\phi_Y(h) = e^{jhb}\phi_X(ah)$.

Q 2.6: Characteristic Function

1. Let X be a random variable with Exponential distribution with mean $\frac{1}{\lambda}$. Find its characteristic function.
2. Let Y be a random variable with Poisson distribution with mean λ. Find its characteristic function.
3. Let U_1 and U_2 be i.i.d. $U[-1,1]$ random variables. Compute the characteristic function of U_1 and of the sum $U_1 + U_2$. Derive the density of $U_1 + U_2$.

Q 2.7: Probability Bounds

Let X be a random variable with Exponential distribution with mean $\frac{1}{\lambda}$. Find an upper bound $\mathbb{P}(X \geq a)$ using Markov's inequality, Chebyshev's inequality and Chernoff bound for some $a > \frac{1}{\lambda}$ and compare with the exact value of $\mathbb{P}(X \geq a)$.

Q 2.8: Convergence in distribution

Let $(W_1, W_2, \ldots)$ be a sequence of i.i.d. random variables with Gaussian distribution with mean 0 and variance σ^2. Let $X_1 = 1$ and $X_{n+1} = \frac{X_n + W_n}{2}$ for $n \geq 1$. Does X_n converge in distribution to some random variable Y? If so, find the distribution of Y.

Q 2.9: Convergence in Probability

Let $\{X_i\}_{i\in\mathbb{N}}$ be a collection of independent and identically distributed random variables with distribution $U[0,\theta]$ (uniform distrubution over $[0,\theta]$). Show that

1. the sequence $\{Y_n\}_{n\in\mathbb{N}}$ with $Y_n = \max_{i=1,2,\ldots,n} X_n$ **converges in probability** to θ.
2. the sequence $\{Z_n\}_{n\in\mathbb{N}}$ with $Z_n = \min_{i=1,2,\ldots,n} X_n$ **converges in probability** to 0.

Q 2.10: Convergence in Distribution

Let $\{X_n\}_{n\in\mathbb{N}}$ be a collection of random variables with

$$\mathbb{P}(X_n = n^2) = \frac{1}{n^2}, \qquad \mathbb{P}(X_n = -1) = 1 - \frac{1}{n^2}, \quad n \geq 1.$$

Show that the distribution of X_n converges as $n \to \infty$. Identify the limiting distribution. Does $\{X_n\}_{n\in\mathbb{N}}$ converge in probability? In mean-square sense?

Q 2.11: Convergence in Distribution

Let $\{X_n\}_{n\in\mathbb{N}}$ be a collection of Geometric random variables with p.m.f.

$$\mathbb{P}(X_n = k) = p_n(1-p_n)^{k-1}, \qquad k \geq 1.$$

where $p_n = \frac{\lambda}{n}$ with $\lambda > 0$. Prove that the sequence $\frac{X_n}{n}$ converges in distribution to an exponential random variable with parameter λ, as $n \to \infty$. Try to show convergence for the tail (complementary C.D.F.) of the distribution.

Q 2.12: Law of Large Numbers

Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with mean μ and standard deviation σ. Then, for any $\epsilon > 0$, show that

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right| > \epsilon\right) = 0.$$

Q 2.13: Application of Central Limit Theorem

Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with uniform distribution over $[0, 4]$. Let $Y = \sum_{i=1}^{100} X_i$. Using (i) Central Limit Theorem and (ii) Chebyshev's inequality, approximate the probability of Y being in range $[180, 220]$.

Q 2.14: Central Limit Theorem in MATLAB

Let $\{X_i\}_{i=1}^{n}$ be a sequence of i.i.d. random variables with $\mathbb{E}[X] = \mu_X$ and $\texttt{var}(X) = \sigma^2$. Let $S_n := \sum_{i=1}^{n} X_i$. Then,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n - n\mu_X}{\sigma\sqrt{n}} \leq x\right) = \Phi(x),$$

where $\Phi(x)$ is the C.D.F. of the Gaussian distribution with mean 0 and variance 1. In other words, S_n is a random variable whose distribution is approximately Gaussian with mean $n\mu_X$ and variance $\sigma\sqrt{n}$. This result is the celebrated Central Limit Theorem (CLT); the proof is standard and can be found in any book on probability theory.

We will visualize this in MATLAB. We will need to use the functions `makedist` and `random`. Familiarize yourself with those functions. Do the following tasks.

1. Define a MATLAB object that represents the Exponential distribution with mean 2. Write a program that generates $N = 1000$ i.i.d. samples from this distribution. Plot the histogram of these samples and see that the plot closely resembles the p.d.f. of exponential distribution. Compute $Z = \frac{S_n - n\mu_X}{\sigma\sqrt{n}}$ corresponding to these samples.
2. Repeat the above process $M = 1000$ times. Store all the Z values and plot the histogram. See that the histogram of Z closely resembles the p.d.f. of the Gaussian distribution with mean 0 and variance 1.

3. Repeat the above steps for each combination of $N = 10, 100, 500$ and $M = 10, 100, 500$.
4. Repeat the above with Beta distribution having parameters 1 and 2, Poisson distribution with parameter $\lambda = 5$ and Rayleigh distribution with parameter 2.

You may also use Python instead of MATLAB.

Q 2.15: Jointly Gaussian Random Variables

Find two random variables X_1 and X_2 that are individually Gaussian, but not jointly Gaussian.

Q 2.16: Gaussian Random Vectors

Let X be a Gaussian random vector on $\mathbb{R}^n$. Show that $Y = AX + b$ is a Gaussian random vector when $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Q 2.17: Gaussian Random Vectors

Let X be a Gaussian random vector with mean $\mu_X \in \mathbb{R}^n$ and covariance matrix C_X. Let $Y = CX + V$ where $C \in \mathbb{R}^{m \times n}$ is a given matrix and V is a zero-mean Gaussian random vector with covariance matrix C_V. Is Y a Gaussian random vector? Is the above true when V has non-zero mean?

Q 2.18: Midsem Autumn 22-23

Prove the following using axioms of probability.

1. For any two events A and B, $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$.
2. Generalize the above to n events: for a collection of events $A_1, A_2, \ldots, A_n$, prove that

$$\mathbb{P}(\cap_{i=1}^n A_i) \geq \sum_{i=1}^n \mathbb{P}(A_i) - (n-1).$$

Q 2.19: Midsem Autumn 22-23

Let Θ be a uniformly distributed random variable over the range $[0, 2\pi]$.

1. Let $X = \cos(\Theta)$ and $Y = \sin(\Theta)$. Are X, Y correlated?
2. Let $X = \cos(\frac{\Theta}{4})$ and $Y = \sin(\frac{\Theta}{4})$. Are X, Y correlated?

Q 2.20: Midsem Autumn 22-23

Let X be a random variable with $\mathbb{E}[X] = 0$ and $\texttt{VaR}[X] = \sigma^2 < \infty$. Prove that for any $a > 0$,

$$\mathbb{P}[X \geq a] \leq \frac{\sigma^2}{\sigma^2 + a^2}.$$

[**Hint:** One approach is to apply Markov's inequality to a random variable $Y = (X + u)^2$ for some u and optimize the bound and relate it to the problem.]

Q 2.21: Midsem Autumn 22-23

This problem is about convergence of random variables.

1. Let $\{X_1, X_2, \dots X_n, \dots\}$ be a sequence of random variables defined as

$$X_n = \begin{cases} \frac{1}{2}\left(1 - \frac{1}{n}\right), & \text{with probability} \quad 0.5, \\ \frac{1}{2}\left(1 + \frac{1}{n}\right), & \text{with probability} \quad 0.5. \end{cases}$$

   Does the above sequence converge to a limiting random variable X^* in the mean-square sense? If so, find X^* and prove its convergence.

2. Let $\{Y_1, Y_2, \dots Y_n, \dots\}$ be a sequence of random variables defined as $Y_n = \frac{(-1)^n U}{n}$ where U is a random variable which is uniformly distributed over $[0, 1]$. Does this sequence converge to a limiting random variable Y^* in the mean-square sense? If so, find Y^* and prove its convergence.

Q 2.22: Midsem Autumn 22-23

Consider a store which opens at time $t = 0$. Customers arrive one after the other. No two customers simultaneously. The time between two consecutive arrivals is a random variable denoted by T which is Exponentially distributed with parameter λ (i.e., the density of T is $f_T(x) = \lambda e^{-\lambda x}$). For a given time t_f, determine

1. the probability that the number of arrivals in the interval $[0, t_f]$ is zero. (2 Points)
2. the probability that the number of arrivals in the interval $[0, t_f]$ is one. (2 Points)

Q 2.23: Endsem Autumn 22-23

Let $\{X_1, X_2, X_3, \dots\}$ be a sequence of continuous random variable such that the density of X_n is given by

$$f_{X_n}(x) = \frac{n}{2} e^{-n|x|}.$$

Show that X_n converges in probability to $X^* = 0$.

Q 2.24: Endsem Autumn 22-23

Consider the daily temperature at IIT Kharagpur as a random process $\{X_n\}$ where X_n is the temperature on day n. Let $\{X_n\}$ be a sequence of i.i.d., Gaussian random variables with a mean of 30 degrees and covariance 10.

1. Let $Y_k := \frac{1}{2}[X_{2k-1} + X_{2k}]$. Is the sequence $\{Y_k\}$ i.i.d?
2. Let $Z_k := \frac{1}{2}[X_k + X_{k-1}]$. Is the sequence $\{Z_k\}$ i.i.d?