EE60039 – Autumn Semester 2023-24

Exercise 1: Probability and Random Processes for Signals and Systems

Prof. Ashish Ranjan Hota
Department of Electrical Engineering, IIT Kharagpur

Q 1.1: Sigma Algebra

1. Let $\mathcal{F}$ be a σ-algebra on Ω. Let $A, B \in \mathcal{F}$. Then show that $A \setminus B \in \mathcal{F}$. (Recall that $A \setminus B := \{x \in \Omega | x \in A \text{ and } x \notin B\}$.)
2. Let Ω be a set of outcomes. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be σ-algebras defined on Ω. Then show that $\mathcal{F}_1 \cap \mathcal{F}_2$ is also a σ-algebra on Ω. Is $\mathcal{F}_1 \cup \mathcal{F}_2$ always a σ-algebra? If not, then give an example where it $\mathcal{F}_1 \cup \mathcal{F}_2$ is not a σ-algebra.
3. Let Ω be a set. Let $\mathcal{F}_1, \mathcal{F}_2, \dots$ be a countable collection of σ-algebras defined on Ω. Then show that $\bigcap_{i=1}^{\infty} \mathcal{F}_i$ is also a σ-algebra.

Q 1.2: Probability Measure

Let $B \in \mathcal{F}$ be an event with $\mathbb{P}(B) \neq 0$. Show that the conditional probability $\mathbb{P}(\cdot|B) : \mathcal{F} \to [0, 1]$ is a probability measure (i.e., it satisfied all three axioms).

Q 1.3: Borel Sigma Algebra

The Borel σ-algebra on the set of real numbers $\mathbb{R}$ is defined as $\mathcal{B}(\mathbb{R}) := \sigma\{(-\infty, x] | x \in \mathbb{R}\}$. In other words, it is the σ-algebra generated by all sets of the form $(-\infty, x], x \in \mathbb{R}$. Then show that for any $a, b \in \mathbb{R}$, the intervals (a, b), $[a, b)$, $(a, b]$ and $[a, b]$ lie in $\mathcal{B}(\mathbb{R})$, i.e., these intervals define valid events. Furthermore, the singleton $\{a\}$ is also an event.

Q 1.4: Conditional Probability

Let A, B, and C be events with $\mathbb{P} > 0$. Show the following.

1. If $B \subset A$, then $\mathbb{P}(B|A) \geq \mathbb{P}(B)$.
2. If $A \subset B$, then $\mathbb{P}(B|A) = 1$.
3. If $B \cap C = \phi$, then $\mathbb{P}(B \cup C|A) = \mathbb{P}(B|A) + \mathbb{P}(C|A)$.

Q 1.5: Conditional Probability

Consider a disease that affects one out of every 1000 individuals. There is a test that detects the disease with 99% accuracy, that is, it classifies a healthy individual as having the disease with 1% chance, and a sick individual as healthy with 1% chance. Then,

1. What is the probability that a randomly chosen individual will test positive by the test?
2. Given that a person tests positive, what is the probability that he or she has the disease?

Q 1.6: Conditional Probability

Consider two events A and B with $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$. Event B is said to be *aligned* with event A if $\mathbb{P}(A|B) > \mathbb{P}(A)$, that is, the occurrence of B implies that the occurrence of A is more likely. If B is aligned with A, then

1. is A aligned with B?
2. is B^c not aligned with A, that is, is it true that $\mathbb{P}(A|B^c) < \mathbb{P}(A)$?
3. is B^c aligned or not aligned with A^c?

Q 1.7: Distribution Function

Let $\Omega = [0,1]$ and the σ-algebra $\mathcal{F} = \sigma\{[a,b] : 0 \le a \le b \le 1\}$ which is the smallest σ-algebra generated by all intervals. Let the probability measure be defined as $\mathbb{P}([a,b]) = \mathbb{P}([a,b)) = \mathbb{P}((a,b]) = b - a$. We now define a random variable $X : \Omega \to \mathbb{R}$ such that

$$X(\omega) = \begin{cases} \omega, & \omega \in [0, 0.5], \\ 1, & \omega \in (0.5, 0.75), \\ 2, & \omega = 0.75, \\ 4\omega, & \omega \in (0.75, 1]. \end{cases}$$

Compute and sketch the C.D.F. of X.

Q 1.8: Continuous Random Variable

Let X be a continuous random variable with density $p_X(x) = C(x - x^2)$ where $x \in [a,b]$ and $C > 0$. What are the possible values of a and b? What is the value of C? Compute the expectation and variance of X.

Q 1.9: Poisson Distribution

Let X be a Poisson random variable with parameter $\lambda > 0$, that is, X takes integer values $0, 1, 2, \ldots$, with $\mathbb{P}(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$. Compute the mean and variance of X. Find the characteristic function of X.

Let $\{X_1, X_2, \ldots, X_n\}$ be independent Poisson random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. Then show that the random variable $Y = \sum_{i=1}^{n} X_i$ is a Poisson random variable. (Hint: Try to compute the characteristic function of Y.)

Q 1.10: Collection of Random Variables

Let $\{X_i\}_{\{i=1,2,\ldots\}}$ be a sequence of independent discrete random variables. Each $X_i \in \{0, 1\}$ with p.m.f. given by

$$\mathbb{P}(X_i = 1) = c^i, \quad \mathbb{P}(X_i = 0) = 1 - c^i,$$

where c is a constant.

(a) What is the range of possible values of c?

(b) Let $Y_k = \sum_{i=1}^{k} X_i$. Find $\mathbb{E}[Y_k]$ and $\lim_{k\to\infty} \mathbb{E}[Y_k]$.

(c) What is the variance of Y_2?

(d) Let $\mathbb{E}[Y_k^2] = \rho_k$. Let $m > k$. Find $\mathbb{E}[Y_k Y_m]$ in terms of ρ_k and c.

(e) Find the characteristic function of Y_k.

Q 1.11: Expectation of Non-negative Random Variables

Show that the expectation of a discrete nonnegative random variable X is $\mathbb{E}[X] = \sum_{x=0}^{\infty} \mathbb{P}(X > x)$. Similarly, $\mathbb{E}[X^m] = \sum_{x=0}^{\infty}((x+1)^m - x^m)\mathbb{P}(X > x)$ where m is an integer.

Q 1.12: Expectation and Moments

Let X and Y be two random variables with zero mean, and $\texttt{var}(X) = 64, \texttt{var}(X+Y) = 68$ and $\texttt{var}(X-Y) = 132$. Find the correlation $\mathbb{E}[XY]$.

Q 1.13: Expectation and Moments

Let X_0 and $W_0, W_1, \ldots$ be random variables that are uncorrelated. Let $\mathbb{E}[X_0] = \mu$, $\mathbb{E}[W_i] = 0$ and $\mathbb{E}[W_i^2] = \sigma_w$ for all $i \in \{0, 1, 2, \ldots\}$. Let

$$X_{k+1} = aX_k + W_k, \quad k \in \{1, 2, \ldots\}.$$

Determine $\mathbb{E}[X_n]$, $\mathbb{E}[X_n^2]$ and $\mathbb{E}[X_n X_m]$ in terms of μ, σ_w, a, n and m.

Q 1.14: Expectation and Moments

Suppose two random variables X and Y have joint pdf

$$f_{XY}(x,y) = \begin{cases} 4xy, & x \in [0,1], y \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Find $\mathbb{E}(X), \texttt{var}(X), \mathbb{E}(Y), \texttt{var}(Y)$ and $\texttt{cov}(X,Y)$.

Q 1.15: Expectation and Moments

Let X and Y be independent random variables each uniformly distributed over the interval $[0,1]$. Let $Z = XY$ (i.e., $Z(\omega) = X(\omega) \times Y(\omega)$ for all $\omega \in \Omega$). Calculate the mean, second moment and variance of Z.

Q 1.16: Expectation and Moments

Two random variables X and Y have mean zero and variances $\sigma_X^2 = 16$ and $\sigma_Y^2 = 36$. Find the variance of $X + Y$ if the correlation coefficient between X and Y is $0, 0.5$ and -0.5, respectively.