Problem 1: let $z_1, z_2 \in Z$. Then $\exists\, x_1 \in X_1,\ x_2 \in X_2$ such that

$$z_1 = a_1 x_1 + a_2 x_2$$

Similarly, $\exists\, y_1 \in X_1,\ y_2 \in X_2$ s.t.

$$z_2 = a_1 y_1 + a_2 y_2$$

Now,

$$\begin{aligned}
\lambda z_1 + (1-\lambda) z_2 &= \lambda (a_1 x_1 + a_2 x_2) + (1-\lambda)(a_1 y_1 + a_2 y_2) \\
&= a_1 (\underbrace{\lambda x_1 + (1-\lambda) y_1}_{\in X_1}) + a_2 (\underbrace{\lambda x_2 + (1-\lambda) y_2}_{\in X_2})
\end{aligned}$$

Hence $\lambda z_1 + (1-\lambda) z_2 \in Z \quad \forall\, \lambda \in [0,1],\ z_1, z_2 \in Z$.

Problem 2: S is epigraph of the function $f(x,y) = x^2 + y^2$, where f is a convex function of $\binom{x}{y}$.

Therefore, S is a convex set.

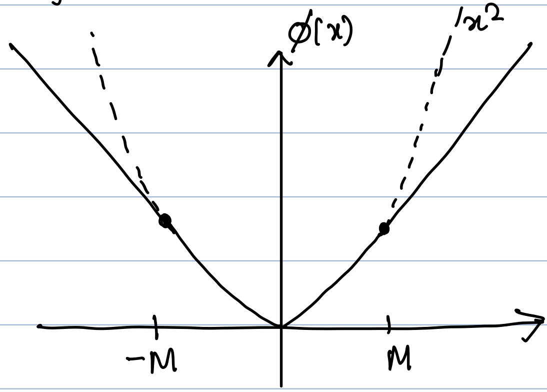
Problem 3:

$$\begin{aligned}
& f(\lambda x_1 + (1-\lambda) x_2) \\
&= h(g(\lambda x_1 + (1-\lambda) x_2)) \\
&\leq h(\lambda g(x_1) + (1-\lambda) g(x_2)) \quad \text{since } g \text{ is concave and } h \text{ is non-increasing} \\
&\leq \lambda h(g(x_1)) + (1-\lambda) h(g(x_2)) \quad \text{since } h \text{ is convex} \\
&= \lambda f(x_1) + (1-\lambda) f(x_2).
\end{aligned}$$

The above holds for all $x_1, x_2 \in \operatorname{dom} f$ and $\lambda \in [0,1]$.

Hence, f is a convex function

Problem 4: The function is convex.



need to show that

$$f(y) \geq f(x) + \nabla f(x)^T (y-x)$$

$\forall\, x, y$.

If both $|x|, |y| \leq M$

or $x, y > M$ or $x, y < -M$, then easy.

Need to show the above holds when $|x| \le M$ and $|y| > M$.
Use the fact that $\nabla f(x)$ is monotonically increasing in x.

Problem 5: any $x \in [a,b]$ can be written as $\lambda a + (1-\lambda)b$.

$$f(x) = f(\lambda a + (1-\lambda)b) \le \lambda f(a) + (1-\lambda) f(b)$$
$$\le \lambda \max(f(a), f(b)) + (1-\lambda)\max(f(a), f(b))$$
$$= \max(f(a), f(b)).$$

Problem 6: The given problem is equivalent to

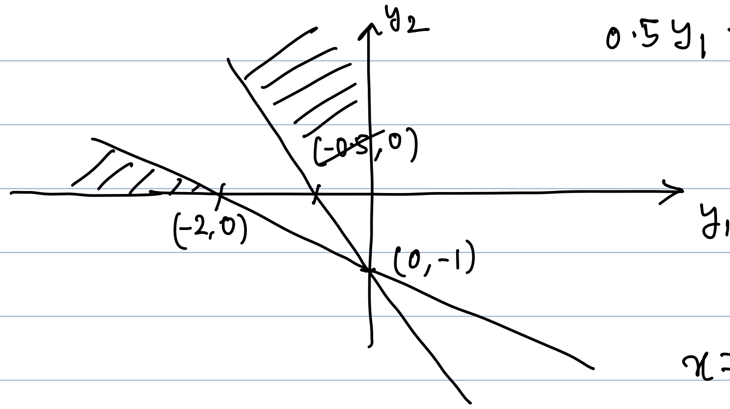
$$\min_x \; c^T x \quad \text{s.t. } Ax = b,\; x \ge 0$$

with $c^T = [0, 1, -1]$

$$A = \begin{bmatrix} 0 & -2 & 0.5 \\ -1 & -1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Dual: $\max_{y \in \mathbb{R}^2} \; y_1 + y_2$

s.t. $A^T y \le c \iff \left.\begin{array}{l} -y_2 \le 0 \\ -2y_1 - y_2 \le 1 \\ 0.5y_1 + y_2 \le -1 \end{array}\right\}$ infeasible



- Dual is infeasible.
- Since primal is feasible, it is unbounded.

$x = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ is feasible to primal.

Problem 7: The given problem is equivalent to

$$\min_{x \in \mathbb{R}} \; 1 + x^2 \quad \text{s.t. } x^2 \ge 1$$

$\Rightarrow$ optimal sol^n is $x^* = 1$, opt. value $= 2$

$$\mathcal{L}(x,\lambda) = 1+x^2 + \lambda(1-x^2)$$
$$= 1+\lambda + x^2(1-\lambda)$$

$$\inf_x \mathcal{L}(x,\lambda) = \begin{cases} -\infty & \text{if } 1-\lambda < 0 \\ 1+\lambda & \text{if } 1-\lambda \geq 0 \end{cases}$$

<u>dual</u>:
$$\max_{\lambda} \quad 1+\lambda$$
$$\text{s.t.} \quad \left.\begin{matrix} 1-\lambda \geq 0 \\ \lambda \geq 0 \end{matrix}\right\} \quad 0 \leq \lambda \leq 1$$

optimal dual solution : $\lambda^* = 1$

optimal dual value $= 2$

strong duality holds.