EE61012 – Spring Semester 2024-25

Homework 3: Convex Optimization in Control and Signal Processing

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Q 3.1: ML Estimation

Let Y be a randon variable with Poission distribution having parameter μ. The probability mass function of Y is given by $\mathbb{P}(Y=k) = \frac{e^{-\mu}\mu^k}{k!}$ for $k = 0, 1, 2, \ldots$. Let $\mu = \sum_{i=1}^{k}(\theta_i u_i) + \theta_{k+1}$ where u_i are features/contexts and $\theta \in \mathbb{R}^{k+1}$ are the unknown weights. We have access to N samples in the form of $(\widehat{u}^i, \widehat{y}^i)_{i \in \{1,2,\ldots,N\}}$. Formulate the problem of finding maximum likelihood estimate of θ as an optimization problem, and determine if the problem is convex.

Q 3.2: GD for Strongly Convex Functions

Recall that the gradient descent algorithm for an unconstrained convex optimization problem is given by

$$x_{t+1} = x_t - \eta_t \nabla f(x_t).$$

Let the cost function f be α-strongly convex and $||\nabla f(x)|| \leq G$ for all x. Then, if $\eta_t = \frac{1}{\alpha(t+1)}$, then we have

$$\frac{1}{T}\sum_{t=0}^{T-1} f(x_t) - f(x^\star) \leq \frac{G^2 \log(T)}{2T\alpha}.$$

Hint: Use potential function $\Phi_t = \frac{t\alpha}{2}||x_t - x^\star||^2$.

Let $\lambda_t := \frac{2t}{T(T+1)}$ and $\bar{x}_t := \sum_{t=1}^{T} \lambda_t x_t$. Then, show that under the above algorithm, we have

$$f(\bar{x}_T) - f(x^\star) \leq \frac{G^2}{\alpha(T+1)}.$$

Q 3.3: Projection is Non-Exapansive

Let $\Pi_X(y)$ denote the projection of a point y on a set X. Show that

$$||\Pi_X(y_1) - \Pi_X(y_2)||_2 \leq ||y_1 - y_2||_2.$$

Q 3.4: Smoothness, Endsem 2023-24

Let $(f_i)_{i \in \{1,2,\ldots,n\}}$ be a collection of n convex functions with f_j being smooth with parameter β_j. Prove that $f(x) = \frac{1}{n}\sum_{j=1}^{n} f_j(x)$ is smooth with parameter $\frac{1}{n}\sum_{j=1}^{n}\beta_j$.

Q 3.5: Gradient Descent, Endsem 2023-24

Consider the following optimization problem:

$$\begin{aligned} \text{minimize}_{x\in\mathbb{R}^3} \quad & \sum_{i=1}^{3} x_i \log x_i + \sum_{i=1}^{2} x_i^2 \\ \text{subject to} \quad & ||x||_2 \leq 3. \end{aligned}$$

Let $x_0 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Carry out two steps of projected gradient descent with step size 0.5.

Q 3.6: Least Squares Regression

A dataset containing 100 samples in the form $\{x^i, y^i\}_{i\in[N]}$ where each $x^i \in \mathbb{R}^3$ and $y^i \in \mathbb{R}$ is given on the website. The output y is a polynomial of degree at most 2 of the input x. Answer the following questions.

1. Define a suitable feature map $\phi(x)$ which maps $x \in \mathbb{R}^3$ to entries of a polynomial of degree 2 in x. What is the dimension of $\phi(x)$?
2. Formulate a least squares problem to determine the coefficients of this polynomial. Clearly state the decision variable w, its dimension, and the cost function.
3. Compute the gradient of this cost function with respect to the decision variable w.
4. Find the optimal weights $w^\star$ using a suitable solver.
5. Compute the error vector $y^i - \phi(x^i)^\top w^\star$ and plot its histogram.
6. Solve the above problem using Gradient Descent, Accelerated Gradient Descent and Stochastic Gradient Descent for 10000 steps. Plot the value of the cost function computed at each iteration and the error $||w_t - w^\star||$ vs. number of interations for all three algorithms.

Q 3.7: Stability and State Feedback Control

Consider the discrete-time dynamical system given by

$$x(k+1) = Ax(k) + Bu(k), \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad x_0 = \begin{bmatrix} -15 \\ 12 \\ -9 \end{bmatrix}.$$

1. Formulate a suitable LMI to determine if 0 is a stable equilibrium point when $u(k) = 0$.
2. Formulate a suitable LMI to determine a static state feedback gain matrix under which 0 is a stable equilibrium point.

Q 3.8: Observer Design and State Reconstruction

Suppose the states are not available to the controller, rather the measured output is given by

$$y(k) = Cx(k), \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Formulate a suitable LMI to an observer gain matrix so that estimation error decays to 0.

Q 3.9: Optimal Control

Let $T = 4$ be the time-horizon. Implement the following optimal control problem for the above dynamics and initial state.

$$
\begin{aligned}
\text{minimize}_{x,u} \quad & \sum_{k=0}^{T-1} \left[||x(k+1)||_2^2 + 0.1u(k)^2 \right] \\
\text{subject to} \quad & x(k+1) = Ax(k) + Bu(k), \\
& -5 \le u(k) \le 5, \\
& \begin{bmatrix} -25 \\ -25 \\ -25 \end{bmatrix} \le x(k+1) \le \begin{bmatrix} 25 \\ 25 \\ 25 \end{bmatrix}, \quad k = 0, 1, \ldots, T-1.
\end{aligned}
$$

Determine if the states are at the equilibrium state $T = 4$. Now implement the following *receding horizon control* scheme. At each step k:

1. Treat $x(k)$ as the initial state and solve the above problem.
2. Once you solve it, apply only the first input computed by the optimization solution ($u(0)$ in the above notation, but actually $u(k)$), and let the state evolve to $x(k+1)$ following the dynamics.
3. Then repeat from $k = 0$ to $k = 20$.

Answer the following questions for the *receding horizon control* scheme.

1. Plot the actual trajectory of the state and control input with respect to time in three different plots with proper labels and axis marks.
2. Compare the above trajectory with the trajectory obtained under static state feedback control law. Does the latter respect the bounds on control input and states?

Q 3.10: LMI Problem (Endsem, Spring 2022-23)

Express the problem of finding a matrix A with $||A|| \le \gamma$ as a linear matrix inequality. Here $||A|| := \sqrt{\lambda_{\max}(A^\top A)}$ where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue of a matrix.

Q 3.11: LMI Problem (Endsem, Spring 2023-24)

Show that the following optimization problem

$$
\begin{aligned}
\text{minimize}_{x \in \mathbb{R}^m} \quad & \frac{1}{2} x^\top P x + q^\top x + r \\
\text{subject to} \quad & F(x) \preceq 0,
\end{aligned}
$$

where P is symmetric, positive definite and $F(x)$ is a LMI is equivalent to

$$
\begin{aligned}
\text{minimize}_{x \in \mathbb{R}^n, t \in \mathbb{R}} \quad & t \\
\text{subject to} \quad & F(x) \preceq 0, \\
& \begin{bmatrix} q^\top x + r - t & x^\top \\ x & -2P^{-1} \end{bmatrix} \preceq 0.
\end{aligned}
$$

Q 3.12: Continuous-time Gradient Flow (Endsem, Spring 2022-23)

Consider the continuous time gradient descent scheme $\dot{x} = -\nabla f(x)$ used to minimize a strongly convex function $f(x)$. The optimal solution $x^\star$ is an equilibrium of the above dynamics. For each of the following functions, determine whether it can be used as a Lyapunov function to prove asymptotic stability of $x^\star$.

1. $V_1(x) = \frac{1}{2}||\nabla f(x)||_2^2$.
2. $V_2(x) = \frac{1}{2}||x - x^\star||_2^2$.
3. $V_3(x) = f(x) - f(x^\star)$.