Lecture-21 : 27th Feb.

Thus far, decision variable x is assumed to take value in $\mathbb{R}^n$.
In some applications, we require x to take integer values.
Those problems are called integer programs.
In some special cases, we require $x \in \{0,1\}^n$,

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, each $x_i \in \{0,1\}$.

$\frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} =$

In general, this class of problems is not convex, and possibly NP-hard.

We will look at a specific subclass of these problems that are "easy" to solve.

Integer Linear Programs :

$$\begin{aligned} \min_{x} \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \\ & x \in \{0,1\}^n \end{aligned} \quad \text{--- (1)}$$

Relaxed version :

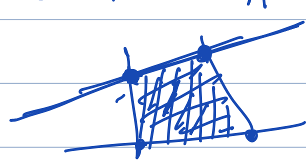
LP-relaxation of --(1)

$$\begin{aligned} \min_{x} \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \\ & x \in [0,1]^n \iff 0 \le x_i \le 1, \ \forall i = 1,2..n \end{aligned} \quad \text{-- (2)}$$

Let p^* be the optimal value of (1)
Let $\tilde{p}^*$ be the optimal value of (2). $\Rightarrow \tilde{p}^* \le p^*$

Can we identify conditions under which $\tilde{p}^* = p^*$?

Equality holds if the matrix A is a totally unimodular matrix (TUM).



Theorem :- Suppose the matrix A is integral, i.e., $A_{ij} \in \mathbb{Z}$. This matrix is TUM if and only if for integral vector b, all the extreme points of the polyhedron $\{x \mid Ax \le b, x \ge 0\}$ are integral.

Q. Given an integral matrix A, how to check if it is TUM?

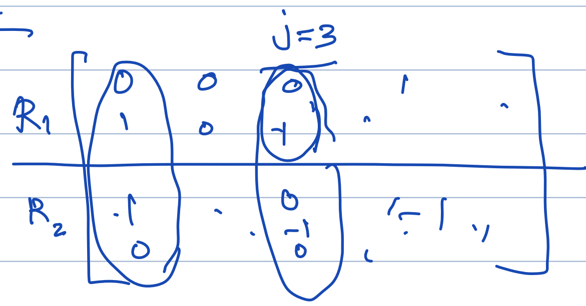
Proposition :

i) Suppose all elements of A are either 0, or 1, or -1.

ii) each column of A has at most two nonzero elements.

iii) the rows of A can be divided into two subsets, denoted by R_1 and R_2, such that

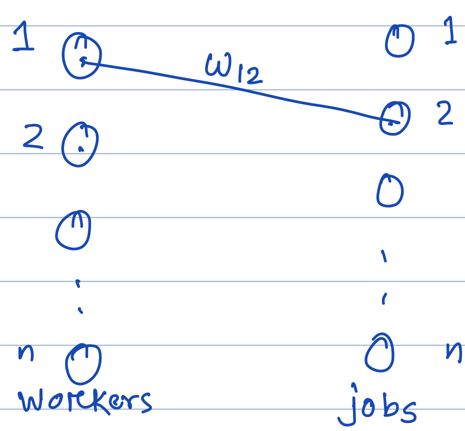
$$\sum_{i \in R_1} a_{ij} = \sum_{i \in R_2} a_{ij} \quad , \text{ for every } j .$$



Minimum cost Perfect Matching

Let w_{ij} : cost of assigning worker i to job j

Goal : Find a matching between workers and jobs to minimize the total cost.



Decision variable : $x \in \{0,1\}^{n^2}$ $\quad x_{ij} = \begin{cases} 1, & \text{if worker } i \text{ is matched to job } j \\ 0, & \text{otherwise.} \end{cases}$

cost function : $f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} x_{ij}$

Constraints : every worker i : $\sum_{j=1}^{n} x_{ij} = 1 \quad \forall i$

every job j : $\sum_{i=1}^{n} x_{ij} = 1 \quad \forall j$

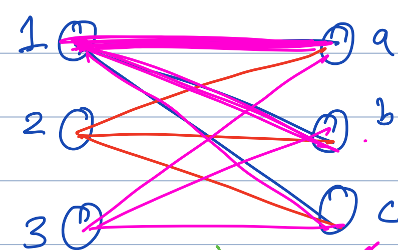
Now, let us express the above constraints as $Ax = b$.

$A \in \mathbb{R}^{2n \times n^2}$

$b \in \mathbb{R}^{2n}$

$x \in \mathbb{R}^{n^2}$

e.g.:- $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$



$$A = \begin{array}{c} 1 \\ 2 \\ 3 \\ a \\ b \\ c \end{array} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}_{2n \times n^2} \begin{matrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{matrix}$$

$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

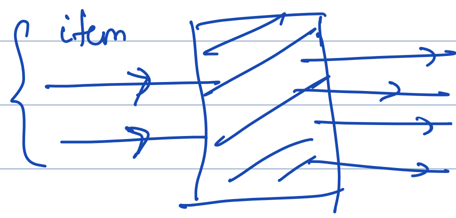
Thus, A is a TUM.
Hence, the polyhedron $\{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ has integral extreme points.

Thus, we can solve the following Linear Program to obtain the optimal assignment / matching.

$$\left[\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\, x_{ij} \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_{ij} = 1, \quad \forall j \\ & \sum_{j=1}^{n} x_{ij} = 1 \quad \forall i \\ & x \geq 0 \end{aligned}\right|$$

Lecture - 22 Inventory Control Problem

Consider a warehouse where we are storing an item.



Let $x(K)$ denote the units stored at time K.

Let $u(K)$ denote amount purchased at time K.

Let $c(K)$: cost of the item ''

Given : $\bar{w}_{0:T-1} = \{ w(0), w(1), \ldots w(T-1) \}$

$x(0)$: units stored at present.

Let p : be the penalty for not fulfilling unit demand
h : cost of storing the item in the warehouse

decision variables : $u_{0:K-1} = \{ u(0), u(1), --- u(T-1) \}$

amount stored at K+1 : $x(K+1) = x(K) + u(K) - w(K)$

$$= x(0) + \sum_{l=0}^{K} u(l) - w(l)$$

constraints : $0 \leq u(K) \leq M \quad , \quad \forall K$

cost function : $\underbrace{\sum_{k=0}^{T-1} c(k)\,u(k)}_{\text{cost of purchase}} + \max\left(h\,x(k+1),\ p(-x(k+1))\right)$

if $x(k+1)$ is positive, we pay the cost of storage.

If $x(k+1)$ is negative, we pay penalty for unmet demand.

$x(0)$ and $w_{0:T-1}$ are known.

$$\min_{\substack{u_{0:T-1}\\ x_{1:T}}} \sum_{k=0}^{T-1} \left[u(k)\,c(k) + \max\left(h\,x(k+1),\ -p\,x(k+1)\right)\right]$$

s.t. $x(k+1) = x(0) + \sum_{l=0}^{k} \left[u(l) - w(l)\right]$, $k = 0, 1, \ldots T-1$

$0 \le u(k) \le M$, $k = 0, 1, \cdots T-1$

$c + \max(a,b) \le t$
$\Updownarrow$
$c + a \le t$
$c + b \le t$

We can further simplify the above using the epigraph form.

$$\min_{u_{0:T-1},\,t_{0:T-1}} \sum_{k=0}^{T-1} t(k)$$

s.t.

(a) $u(k)\,c(k) + h\left[x(0) + \sum_{l=0}^{k} \left(u(l) - w(l)\right)\right] \le t(k)$

(b) $u(k)\,c(k) - p\left[x(0) + \sum_{l=0}^{k} \left(u(l) - w(l)\right)\right] \le t(k)$

$0 \le u(k) \le M$

all constraints hold for $k = 0, 1, 2 \ldots T-1$.

Q) What happens when $w_{0:T-1}$ is not known with certainty?

Let us assume that each $w(l) \in \left[w^l_{lb},\ w^l_{ub}\right]$.

to solve the above problem, constraints (a) and (b) need to hold for all $w(l) \in \left[w^l_{lb},\ w^l_{ub}\right]$.

for (a): $g - \sum h\, w(\ell) \leq t(\ )$ $\forall\, w(\ell) \in [w_{lb}^{\ell}, w_{ub}^{\ell}]$

for (b): $g + p \sum_{\ell} w(\ell) \leq t(k)$

if the constraint holds for all $w(\ell) = w_{ub}^{\ell}$, then it holds for all $w(\ell) \leq w_{ub}^{\ell}$ as well.

The robust version of the problem is given by:

$$\min_{u_{0:T-1},\, t_{0:T-1}} \sum_{k=0}^{T-1} t(k)$$

s.t.

(a) $u(k)\, c(k) + h \left[x(0) + \sum_{\ell=0}^{k} (u(\ell) - w_{lb}^{\ell})\right] \leq t(k)$

(b) $u(k)\, c(k) - p \left[x(0) + \sum_{\ell=0}^{k} (u(\ell) - w_{ub}^{\ell})\right] \leq t(k)$

$0 \leq u(k) \leq M$

all constraints hold for $k = 0, 1, 2 \ldots T-1$.

non-adaptive "open-loop" strategies

Ideally,

$$u(k) = \bar{u} + \sum_{\ell=0}^{k-1} a_{\ell k}\, w(\ell)$$

feedback policies

adaptive policies are going to improve the optimal value in a significant manner.

Statistical Estimation

Y: observation, θ: parameter, unknown.

we know $f_Y(y;\theta)$ is known.

Suppose $\hat{y}$ is our observation.

likelihood function $\mathcal{L}(\theta) = f_Y(\hat{y};\theta)$

maximum likelihood estimate $\max_{\theta} \mathcal{L}(\theta) = \max_{\theta} f_Y(\hat{y};\theta)$

Suppose we have N i.i.d observations: $(\hat{y}_1, \hat{y}_2, \ldots \hat{y}_N)$

likelihood: $\mathcal{L}(\theta) = f_Y(\hat{y}_1;\theta)\, f_Y(\hat{y}_2;\theta) \cdots f_Y(\hat{y}_N;\theta)$

$$\log \mathcal{L}(\theta) = \sum_{i=1}^{N} \underbrace{\log(f_Y(\hat{y}_i;\theta))}$$

$$\max_{\theta} \mathcal{L}(\theta) \equiv \min_{\theta} -\log(\mathcal{L}(\theta))$$

$$= \min_{\theta} \sum_{i=1}^{N} \left[-\log(f_Y(\hat{y}_i;\theta))\right]$$

Special cases: $y = a^T x + v$, $v \sim \mathcal{N}(\mu, \sigma^2)$

a : known vector

Given $(\hat{y}_1, \hat{y}_2, \ldots \hat{y}_N)$.

x : unknown parameter we wish to find.

μ, σ^2 : known

Let us first determine

$$f(\hat{y}_1; x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\hat{y}_1 - a^T x - \mu)^2}{2\sigma^2}\right).$$

$$-\log f(\hat{y}_1; x) = (\text{const}) + -\left(-\frac{(\hat{y}_1 - a^T x - \mu)^2}{2\sigma^2}\right)$$

$$= (\text{const}) + \frac{1}{2\sigma^2}\left(\hat{y}_1 - a^T x - \mu\right)^2$$

$$\frac{1}{2\sigma^2}\sum_{i=1}^{N}\left(\hat{y}_i - \mu - a^T x\right)^2 + \text{cost.}$$

ML estimate of x: $\quad \underset{x}{\text{argmin}} \sum_{i=1}^{N} \left(\hat{y}_i - \mu - a^T x\right)^2$

→ simply a least sq. problem.

If v has Laplace distribution: $f_v(v) = \frac{1}{\sqrt{2\alpha}} \exp\left(-\frac{|v|}{\alpha}\right)$,

α is a parameter which is known.

Likelihood of observing $\hat{y}_1$: $\quad \frac{1}{\sqrt{2\alpha}} \exp\left(-\frac{|\hat{y}_1 - a^T x|}{\alpha}\right)$.

$$-\log \mathcal{L}(x) = \left(-\log \frac{1}{\sqrt{2\alpha}}\right) + \frac{1}{\alpha}\left|\hat{y}_1 - a^T x\right|$$

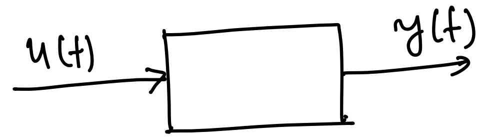
Overall ML estimation problem:

$$\hat{x}_{ML} = \underset{x}{\text{argmin}} \sum_{i=1}^{N} \left|\hat{y}_i - a^T x\right|$$

→ another regression problem, now w.r.t. 1-norm.

Module C: Convex Optimization in Control

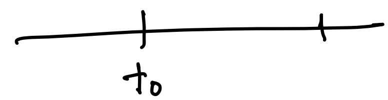
In a nutshell, control theory is the study of influencing trajectories of a dynamical system to satisfy desired properties.



- Static vs. Dynamic System:

Static: output at time τ is a function of input at time τ, and not at any other time

Dynamic: output $y(t)$ potentially depends on $u(\tau)$, $t \neq \tau$.



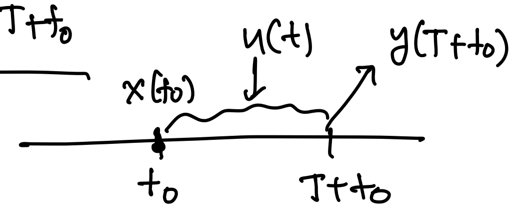
- Example:

m: mass of the object

$p(t)$: position of the object
$v(t)$: velocity
$F(t)$: force applied to it

$$\frac{d^2 p(t)}{dt} = \frac{F(t)}{m}$$

Given a causal dynamical system, a variable $x(t)$ is called a state variable if knowledge of $x(t_0)$ and input $(u(t))_{t_0 \leq t \leq T+t_0}$ is sufficient to determine its output $y(T+t_0)$.



- State-space representation:

$$\left.\begin{aligned} \frac{dx(t)}{dt} = \dot{x}(t) &= f(x(t), u(t), t) \\ y(t) &= h(x(t), u(t), t) \end{aligned}\right] \text{CT.}$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1(x, u, t) \\ f_2(x, u, t) \\ \vdots \\ f_n(x, u, t) \end{bmatrix}$$

Example: $x_1(t) = p(t)$, $u(t) = F(t)$
$x_2(t) = v(t)$

$$\begin{bmatrix} \dot{x}_1(t) = \frac{dp(t)}{dt} = v(t) = x_2(t) \\ \dot{x}_2(t) = \frac{F(t)}{m} \end{bmatrix} \quad \begin{aligned} f_1 &= x_2 \\ f_2 &= u/m \end{aligned}$$

$y(t)$: quantities that we observe / measure, usually with a sensor.

For a discrete-time system, $\quad x_{k+1} = f(x_k, u_k, k)$

$k = 0, 1, 2, \ldots \quad y_k = h(x_k, u_k, k)$

For a system $\dot{x} = f(x,u)$, the pair $(\bar{x},\bar{u})$ is said to be an eq^m point if $f(\bar{x},\bar{u}) = 0$

Questions of Interest

$\Rightarrow \dot{x} = 0$.

- Stability: Given an equilibrium point, can we determine if it is stable?

- Identification: Can we determine unknown parameters governing the system dynamics from input-output data?

- State Estimation : Suppose $y(t) \neq x(t)$. Can we estimate the states from input-output data?

- Optimal Control: Can we steer the states/output to a desired/reference state/output while spending minimal control effort?

- Robust Control: Can we design a controller that achieves desired performance for a range of values of unknown parameters?

We will see that many of the above problems can be formulated and solved using convex optimization.

A. Discrete-time Optimal Control

given: $x_{k+1} = f(x_k, u_k, k)$

- Discrete-time State-Space Model: let $y_k = x_k$

$\dot{x}(t) = f(x(t), u(t))$ can be discretized using Euler discretization as $x_{k+1} = x_k + h \cdot f(x_k, u_k)$

- Goal: Starting from an initial state z_0, compute a sequence of control inputs $(u_0, u_1, \ldots, u_T)$ such that the state at time T, denoted $z_T = z^{\text{des}}$ which is the desired state.
- Let us formulate an optimization problem to achieve this goal.
- Decision Variables: $(u_0, u_1 \ldots u_T, x_0, x_1 \ldots x_T) = \bar{X}$
- Cost Function:

$$f(\bar{X}) = \begin{cases} \underbrace{\| x_T - z^{des} \|_2^2} \\ \sum_{k=1}^{T} \| x_k - z^{des} \|_2^2 \\ \sum_{k=1}^{T} \left[\| x_k - z^{des} \|_2^2 + \underbrace{\lambda \| u_k \|_2^2}_{\text{control effort, } \lambda > 0 \text{ : constant.}} \right] \end{cases}$$

- Constraints:

$x_{k+1} = f(x_k, u_k, k), \quad k = 0, 1 \ldots T$

$x_0 = z_0$

$u_k^{min} \leq u_k \leq u_k^{max}$

$g(x_k, u_k) \leq 0$: application specific constraint.

Finite-Horizon Optimal Control Problem

$$\min_{\substack{u_0, u_1, \dots, u_T \\ x_0, x_1 \dots x_{T+1}}} \sum_{k=0}^{T} \left[\|x_{k+1} - z^{des}\|_2^2 + \lambda \|u_k\|_2^2 \right]$$

$$\text{s.t.} \quad x_{k+1} = f(x_k, u_k, k), \quad k = 0, 1, 2 \dots T$$

$$u_k^{min} \le u_k \le u_k^{max}, \quad ''$$

$$g(x_k, u_k) \le 0, \quad ''$$

$$x_0 = z_0.$$

When is the above problem a convex optimization problem?

- cost function is convex, when $\lambda > 0$.
- g needs to be a convex function of (x, u).
- function f needs to be affine, i.e.,

$$x_{k+1} = A_k x_k + B_k u_k + f_k \quad \text{for some known } f_k.$$

i.e., the dynamical system needs to be a linear system.

Discrete-time Linear Quadratic Regulation Problem

$$
\left[\begin{array}{ll}
\min\limits_{\substack{u_0 \dots u_T \\ x_0 \dots x_{T+1}}} & \sum_{k=1}^{T} \left[x_{k+1}^T Q x_{k+1} + u_k^T R u_k \right] \\
\text{s.t.} & x_{k+1} = A x_k + B u_k , \quad k = 0, \dots T \\
 & x_0 : \text{given/known constant}
\end{array}\right.
$$

$$
\left[u_k = K x_k \right]
$$

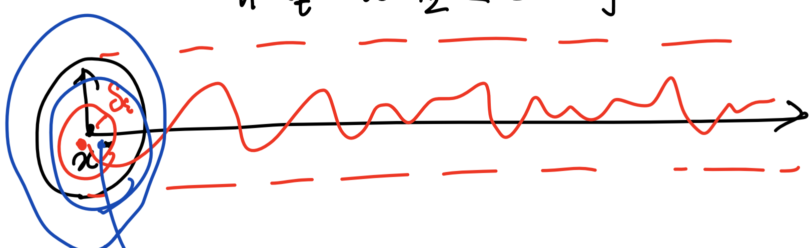
B. Stability

- Consider a continuous-time (autonomous) dynamical system: $\dot{x} = f(x)$ with initial state x_0.

- Equilibrium point: x^* is an eq^m point if $f(x^*) = 0$.

- Stability of an equilibrium point: x^* is said to be stable if

for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that if $\|x_0 - x^*\|_2 \le \delta_\varepsilon$, then $\|x_t - x^*\|_2 \le \varepsilon$ for all $t \geq 0$.



If $\lim_{t \to \infty} x(t) = x^*$, we say that x^* is asymptotically stable.

x^* is unstable if there exists some $\bar{\varepsilon}$ s.t. we cannot find any $\delta > 0$ satisfying the above property.

- Lyapunov Stability Theorem: