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On Leonardo da Vinci's Cat and Mouse Problem

ABHIJIT GUHA
Whittle Laboratory, University of Cambridge
and SUBIR K. BISWAS
Computer Laboratory, University of Cambridge

An old problem of finding the curve of pursuit and the point of intersection, originally proposed by Leonardo da Vinci, is considered in its generality. It is assumed that the pursued moves along an arbitrary straight line and the motion of the pursuer is always directed towards the pursued. Both the pursuer and the pursued move at constant but arbitrarily different speeds. A novel and elegant scheme of calculation is devised here by which the point of intersection can be calculated very simply without any explicit determination of the trajectory of the pursuer. Then the scheme is extended to give the analytical expressions for the actual curve of pursuit. The present method of solution is more general and very considerably simpler than existing solution techniques. Some physics of the problem is discussed.

A MATHEMATICAL problem which is often quoted in the texts on mechanics and differential equations,[1–5] and which dates back to Leonardo da Vinci concerns the path of a cat chasing a mouse. It is assumed that both of them run at constant speeds, the mouse runs along a straight line and the cat is always directed towards the mouse. In Leonardo's formulation, the straight line path followed by the mouse is at right angles to the line joining their initial positions. The curve of pursuit in this case was first solved by Pierre Bouguer, a French hydrographer, in a paper published in 1732 and the problem was included by George Boole[2] in his book on differential equations in 1859. (An historical account of pursuit problems has been given by Archibald and Manning.[6]) Chorlton[4] solved this problem in a relative frame of reference. He also considered a generalisation of this problem where the mouse's straight-line path could be inclined at an arbitrary angle α (Fig. 1).[5] Colman[7] has recently given a closed-form analytical solution of this generalised problem in the absolute frame of reference. All these previous workers were interested in specifying the *whole* trajectory of the cat and the solution procedures of such an apparently innocuous problem were extremely laborious and complicated (see, for example, reference 7). The coordinate of the point of capture was then calculated as an end result by finding the intersection of this trajectory with that of the mouse. On the other hand, if one is interested only in finding the point of capture, explicit determination of the cat's trajectory (and consequently a horrifying amount of calculus and algebra) can be *by-passed*. In this article we present a simple and elegant method of doing so. Later it is also shown that the present method can easily be extended to determine the curve of pursuit and analytical expressions for the curve can be derived with considerably less effort than Colman's method. Furthermore, the solution presented by Colman is valid only when the cat moves faster than the mouse. The present method of solution is applicable even when the reverse is true.

We consider the general problem like Colman[7] and use a fixed frame of reference. The mouse is initially at point I, the cat is at 0 (Fig. 1) and their initial separation is y_0. Assume that the speeds of the mouse and the cat are v

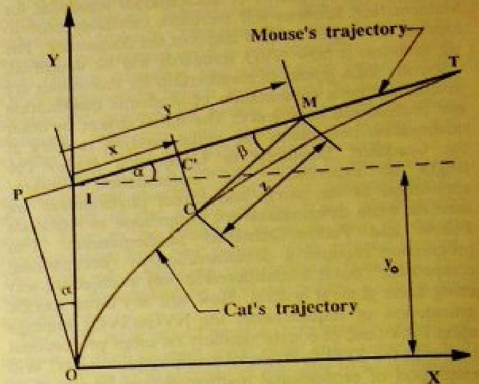

Fig. 1

and kv, respectively and the mouse's path IT is inclined at an angle α to the X-axis. α may take any value between $-90°$ and $90°$ to cover all possible cases. Consider the motion of the cat at an intermediate point such as C. The tangent to the cat's trajectory at C meets IT at an angle β at M which is also the instantaneous position of the mouse (by terms of the problem). Construct CC' perpendicular to IT. We specify the distances IC', IM and CM by the variables x, y and z, respectively. Now by taking the components of the velocities in the appropriate directions we can write

$$\frac{dx}{dt} = kv\cos\beta \tag{1}$$

$$\frac{dy}{dt} = v \tag{2}$$

$$\frac{dz}{dt} = v\cos\beta - kv, \tag{3}$$

where t denotes time. Eliminating $\cos\beta$ from (1), (2) and (3)

$$\frac{dz}{dt} - \frac{1}{k}\frac{dx}{dt} + k\frac{dy}{dt} = 0. \tag{4}$$

Equation (4) may be integrated to give

$$z - \frac{x}{k} + ky = c. \tag{5}$$

At $t = 0$, $z = y_0$, $y = 0$, $x = IP = -y_0\sin\alpha$, therefore

$$c = y_0 + \frac{y_0}{k}\sin\alpha. \tag{6}$$

Note that equation (5) is generally valid, and no restriction is imposed on the speed ratio k. When $k > 1$, capture occurs. At the point of capture T, $z = 0$, $x = y = IT = L_m$ = the distance ran by the mouse. Therefore, from (5),

$$L_m = y_0 \frac{k + \sin\alpha}{k^2 - 1}. \quad (7)$$

The coordinate of the point of capture T can therefore be calculated from simple geometry,

$$X_T = y_0 \frac{\cos\alpha(k + \sin\alpha)}{k^2 - 1}$$

$$Y_T = y_0 \left(1 + \frac{\sin\alpha\,(k + \sin\alpha)}{k^2 - 1}\right). \quad (8)$$

If $\alpha = 90°$ (when the mouse runs directly away from the cat), equation (8) gives $Y_T = y_0 \frac{k}{k-1}$, and if $\alpha = -90°$ (when the mouse runs directly towards the cat for a head-on collision!), $Y_T = y_0 \frac{k}{k+1}$ which is obviously correct. If we put $\alpha = 0$ in equation (8) then we obtain the known result for Leonardo's problem,[1] $X_T = y_0 \frac{k}{k^2-1}$, $Y_T = y_0$. (Also note that, in this case, when $X_T = y_0$, k is equal to the golden ratio in the Fibonacci series.) Equation (8) thus represents the solution of the generalised problem ($k > 1$). If $k = 1$, the path of the cat becomes asymptotic to that of the mouse and the cat *finally* develops a constant lag behind the mouse. The magnitude of this constant lag can easily be found by substituting $k = 1$ and $z = y - x$ in equation (5) and is given by

$$z_{\text{final}} = \frac{y_0}{2}(1 + \sin\alpha). \quad (9)$$

Equation (9) is plotted in Fig. 2. z_{final} in equation (9) reduces to the obvious limits of 0 and y_0 for $\alpha = -90°$ and 90°, respectively.

From the mouse's point of view, it should choose α such that it can run the longest distance before being caught (which would increase, for example, the probability of finding holes randomly distributed on the earth). Equation (7) shows that L_m is maximum when $\alpha = 90°$ which is also intuitively obvious. The maximum value of L_m is $y_0/(k-1)$. Figure 3 plots the distance

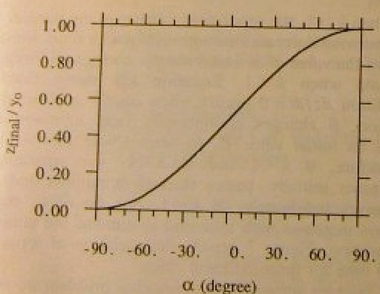

Fig. 2. Final constant lag of the cat when $k = 1$

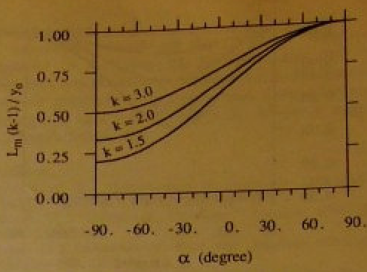

Fig. 3. Normalised escape length of the mouse

covered by the mouse before capture, which is normalised by the factor $y_0/(k-1)$. Thus the different curves in Fig. 3 quantify the relative merits of the different paths (all straight lines) followed by the mouse, subject to the varying speed ratio k.

It should be noted that although Leonardo's cat has the instinctively correct response in always directly chasing the mouse, it could save time (and hence increase the probability of capture) by applying some rudimentary knowledge of mathematics. It could, for example, predetermine the point of capture (assuming, of course, that the mouse still runs along a straight line!) and run in a straight line upto that point. By simple trigonometry it can be shown that the length covered by this mathematical cat is given by,

$$L_{c,\text{math}} = y_0 \frac{k}{k^2 - 1}[\sin\alpha + \sqrt{\sin^2\alpha + k^2 - 1}]. \quad (10)$$

Equation (7) gives the length covered by Leonardo's cat as

$$L_{c,\text{vinci}} = kL_m = y_0 \frac{k}{k^2 - 1}[k + \sin\alpha]. \quad (11)$$

Note that although we have not yet explicitly determined the cat's trajectory, we can find its length between O and T! We can formulate a measure of the coefficient of performance of Leonardo's cat, η_{cat}, by taking the ratio of equations (10) and (11)

$$\eta_{\text{cat}} = \frac{L_{c,\text{math}}}{L_{c,\text{vinci}}} = \frac{[\sin\alpha + \sqrt{\sin^2\alpha + k^2 - 1}]}{k + \sin\alpha}. \quad (12)$$

Figure 4 plots η_{cat} versus α for three different values of k. The curves are not symmetrical about $\alpha = 0°$ and the minimum η_{cat} occurs at some *negative* value of α. For same α, η_{cat} increases with increasing speed ratio k. This is expected because as k increases, the relative importance of the mouse's motion diminishes. Thus in the limit $k \to \infty$, the path taken by Leonardo's cat approaches that of the mathematical cat for all α. Such coincidence occurs for finite k only when $\alpha = \pm 90°$. When $\alpha = -90°$, the mouse has chosen the worst path (Fig. 3) and hence it is not unexpected that it turns out to be the best for Leonardo's cat. However, it is possibly a bit surprising that when $\alpha = 90°$, the mouse has taken the best course of action and simultaneously Leonardo's cat also becomes the most efficient ($\eta_{\text{cat}} = 1$).

Having seen that quite useful information may be obtained from such a simple integral analysis, we now

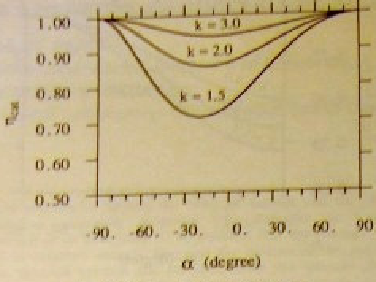

Fig. 4. Efficiency of Leonardo's cat

turn our attention to find the equation for the curve of pursuit. We substitute equations (1), (2) and the identity $z=(y-x)/\cos\beta$ in equation (5) to give

$$\frac{dx}{dy}=\frac{x-y}{y-x/k^2-c/k}. \tag{13}$$

Equation (13) is a *simple first order* differential equation *valid for any* k ($k \gtreqless 1$) and may be readily solved with the initial condition $y=0$, $x=-y_0\sin\alpha$. The solution is (when k is either greater or less than 1)

$$|L_m-y|=\frac{1}{A}\frac{|k-\eta|^{(k-1)/2}}{|k+\eta|^{(k+1)/2}} \tag{14}$$

where

$$\eta=\frac{x-L_m}{y-L_m} \tag{15}$$

$$A^2=\frac{1}{(ky_0)^2}\frac{|k-1|^{(k+1)}}{(k+1)^{(k-1)}}\frac{(1-\sin\alpha)^{(k-1)}}{(1+\sin\alpha)^{(k+1)}}. \tag{16}$$

When $k>1$, $y\leq L_m$ and hence the modulus sign may be removed from the LHS of equation (14). As y varies from 0 to L_m, η varies *monotonically* from $(1+y_0\sin\alpha/L_m)$ to k, between the initial point and the point of capture. (Using equations (1), (2) and L'Hospital's theorem, it can be shown from equation (15) that $\eta\rightarrow k$ as $y\rightarrow L_m$). When α is positive, η is always greater than 1. However, when α is negative, the initial value of η is less than 1. Hence η becomes unity at some intermediate point. Thus, when $\alpha<0$, the condition $x=y$ occurs twice: once when $\eta=1$ (at the point where the curve of pursuit overturns, $\beta=90°$, $z\neq 0$) and then again, as usual, at the point of capture ($z=0$, $\beta=0$). The easiest way of computing the curve of pursuit would be to choose a value of η between its limits. Equation (14) then gives y, equation (15) gives x and equation (5) gives z. x, y and z completely specify the position of the cat in the $X-Y$ plane. Fig. 5 gives two examples of such calculations when $\alpha=45°$, $k=2$ and $\alpha=-45°$, $k=1{\cdot}5$, respectively. Other than the special case of $\alpha=-90°$, Leonardo's cat always catches the mouse from behind! (This is why capture does not occur at any other α, when the speed ratio k is less than unity.)

When $k<1$, η may not vary monotonically and may not always remain finite. Hence it is better to write equation (14) in the form, after factorizing $(y-L_m)$ out from both sides (see Appendix)

$$k(y-L_m)+(x-L_m)$$
$$=A^{-2/(k+1)}|\{k(y-L_m)-(x-L_m)\}|^{(k-1)/(k+1)}. \tag{17}$$

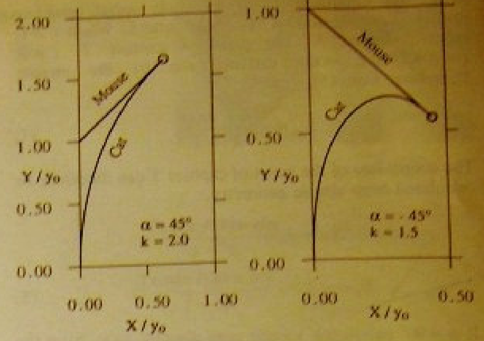

Fig. 5. Analytical solution of curves of pursuit

The value of L_m in the above equation is still given by equation (7), but L_m does not have any physical significance when $k<1$ as capture does *not* occur (except in the special case of $\alpha=-90°$). (When $k>1$, L_m is the distance run by the mouse before being captured.) y always increases monotonically with time since the mouse moves with constant speed. For a certain value of y, x may be calculated from equation (17). Equation (5) again gives z and the coordinate of the cat can be found. Fig. 6 shows an example of the curve of pursuit when the speed ratio k is less than 1. The cat's path ultimately becomes asymptotic to the mouse's path and then their separation increases indefinitely.

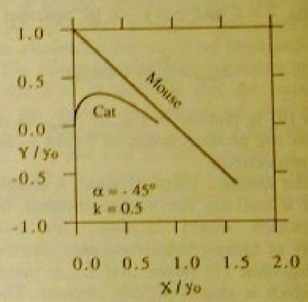

Fig. 6. Analytical solution when $k<1$

When $k>1$, the separation z between the pursuer and the pursued decreases monotonically with time, irrespective of the value of α. Interestingly, z may pass through a minima when $k<1$. Equation (3) shows that the condition $dz/dt=0$ occurs when $\cos\beta=k$. During the motion, β changes continuously (and monotonically) from its initial value β_0 to zero. (Note $\beta_0=90°-\alpha$.) Therefore, if $k<1$ and $\sin\alpha<k$, the separation z decreases initially, passes through a minima and then increases indefinitely. If $k<1$, but $\sin\alpha>k$, then z always increases with time and y_0 remains the minimum separation. Figure 7 depicts the variation of separation with time for three different cases.

A modern version of this historical problem would be, say, the determination of the path of a guided missile following a moving target. As yet no analytical solution

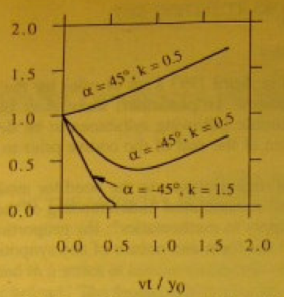

Fig. 7. Variation with time of the separation between the pursuer and the pursued

exists if the pursued moves even along any simple curve other than a straight line. For arbitrary complex paths of the pursued, numerical simulation is perhaps the only method of solution.

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Appendix

Consider the sign of the LHS of equation (17). Introducing (1), (2) and (3), and the shorthand notation D,

$$\frac{d}{dt}[k(y - L_m) + (x - L_m)] = \frac{dD}{dt} = kv(1 + \cos\beta) \geq 0. \tag{18}$$

Therefore, D always increases with time and the sign of D depends on its initial value, D_0. Using (7) and the initial condition: at $t = 0$, $y = 0$, $x = -y_0 \sin\alpha$, we obtain

$$D_0 = D(t = 0) = -y_0 \frac{k}{k-1}[1 + \sin\alpha]. \tag{19}$$

$D_0 < 0$, when $k > 1$; and $D_0 > 0$, when $k < 1$. Thus $D = k(y - L_m) + (x - L_m)$ is always positive when $k < 1$, as suggested by equation (17). (Equation (17), with a negative sign in front of the RHS, can be used when $k > 1$.)