A Simple Analytical Theory for Interpreting Measured Total Pressure in Multiphase Flows

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1 Introduction
Mixtures of gases laden with solid particles or mixtures of a vapor and liquid droplets occur frequently in many practical flow situations. Pilot measurements are often used for inferring velocity or loss (entropy generation) in multiphase mixtures. However, in single phase fluids, it is normally permissible to assume that the fluid is brought to rest at the reaction of the Pitot tube (isentropically). Hence flow Mach number and entropy generation in steady, adiabatic flow are uniquely determined by the total pressure measured by a Pitot tube, together with an independent measurement of the static pressure. In supersonic flow in an ideal gas, application of Rankine-Hugoniot equations across the detached shock wave in front of a Pitot tube retains the utility of Pitot measurements for deducing flow Mach number and entropy generation. Pitot measurements in a multiphase mixture, however, require careful considerations (Guha, 1998).

The solid particles or the liquid droplets respond to changes in temperature, velocity, etc. of the gas phase through interphase exchanges of mass, momentum, and energy. These are essentially rate processes and hence significant departure from equilibrium can take place if the rate of change of external conditions, imposed by the deceleration in the stagnating flow, is comparable to the internal time scales. Thus, for example, if the size of the liquid droplets or the solid particles is very small, then inertial and thermodynamic equilibrium between the two phases are maintained always, and a Pitot tube would measure the equilibrium total pressure, \( p_{eq} \). On the other hand, if the size of the droplets or the particles is very large, all interphase transfer processes remain essentially frozen. The Pitot tube reads the pressure which it would have recorded if the vapour phase alone was brought to rest from the same velocity. The total pressure in this case is termed the frozen total pressure, \( p_{f} \).

As an example, consider low-pressure wet steam with a typical wetness fraction of 10 percent and a Mach number 1.5. Calculations show that \( p_{f}/p_{eq} = 3.3 \) and \( p_{f}/p_{eq} = 3.75 \), where \( p_{eq} \) is the static pressure. Therefore, in this particular example, the equilibrium total pressure is about 15 percent higher than the frozen total pressure.

The subject of interpretation of measured total pressure in two-phase mixtures has been considered by Taylor (1955), Dussourd and Sharpe (1954), Siv (1956), Raddiker (1964), and Crane and Moore (1972). None of these previous authors attempted any generalization for all flow conditions and different two-phase mixtures. For example, Crane and Moore considered Pitot measurements in wet steam at low subsonic velocities and concluded that the total pressure measured would lie between the frozen and the equilibrium value, and would virtually be independent of the droplets size. So, on the other hand, asserted that, for low mass fractions of the dispersed phase, a total head probe in a gas-particles flow would always measure the frozen total pressure.

However, as explained earlier, it is expected that for intermediate sizes of the droplets or particles, the pressure recorded by the probe would neither be the equilibrium nor the frozen value. The imposed deceleration in front of the Pitot tube would render the two-phase mixture into nonequilibrium situations, both kinetically and thermodynamically. The deceleration process consequently ceases to be isentropic, as nonequilibrium exchanges of mass, momentum, and energy between the two phases create entropy. The degree of the nonequilibrium effects will depend on the droplet size, mass concentration, etc. or more precisely on the relaxation times of the system. Hence, relaxation gas dynamics has to be called for a solution of the problem. (For a broad overview of the relaxation gas dynamics in vapor-droplet flow, one may refer to Guha, 1995.)

For a proper solution of the real flow field around the mouth of a Pitot tube, one therefore has to solve multidimensional (at least two dimensional axisymmetric) conservation equations with viscous, and thermal and inertial nonequilibrium effects. It is important to incorporate the inertial nonequilibrium effects, i.e., to allow a velocity slip between the two phases. It is so, not only
because the inertial effects are themselves significant but also because restraining the two phases to travel at the same velocity has serious implications for the thermal equilibration process. For example, a fluid particle moving along the stagnation streamline takes infinite time to reach the stagnation point. Therefore, in the absence of any velocity slip, the vapor—droplet mixture would have long enough time to come to thermal equilibrium, irrespective of the magnitude of the thermal relaxation time. In reality, however, large droplets may move without significant interphase mass and energy transfers and finally hit the back of the probe without influencing the pressure.

Most available computational schemes (explicit time-marching, e.g., Guha and Young, 1991; Young, 1992) however, neglect any velocity slip between the phases. In most situations with submicron size droplets, this is an acceptable assumption and saves computer time tremendously. (The relaxation time governing the velocity slip is very small and hence the computational time step required for numerical simulation is small resulting in large CPU time.) Even if all these effects are incorporated in the computer programs, it is quite likely that the numerical entropy generation would mask the entropy generation due to relaxation processes. Hence Guha (1998) presented a simple, quasi-one-dimensional model in which some plausible assumptions about the variation of the gas phase velocity were made. Step-by-step numerical integration of the conservation equations for a multiphase mixture, between a supersonic point and the stagnation point, then determined the pressure that would have been recorded by a measuring device.

Guha (1998) considered a large number of two-phase mixtures, both gas—particle and vapour—droplet, at subsonic as well as supersonic velocities for many different sizes of the droplets (or particles). In the supersonic case a detached frozen shock wave stands in front of the Pitot tube. The relaxation mechanisms in a gas-particle mixture are different from those in a vapour—droplet flow. Despite all these complexities and differences, it was possible to properly nondimensionalization of flow parameters to adopt a universal plot, within acceptable tolerance, of nondimensional total pressure versus Stokes number (which is a nondimensional representation of particle inertia, Eq. (32)). The predicted total pressure correctly reduced to the frozen total pressure in the limit of large Stokes number (large particles) and to the equilibrium total pressure in the limit of small Stokes number (small particles). Maximum depression of the total pressure on the Stokes number was observed when the Stokes number was of the order unity.

Guha's (1998) calculations were, however, numerical in nature. In this paper, we develop an analytical theory for determining total pressure under nonequilibrium conditions, which is amenable to direct physical interpretation.

Nomenclature

- $A$: cross-sectional area of the stream tube
- $a$: equilibrium speed of sound
- $a_l$: frozen speed of sound
- $c_{p,\text{ed}}$: specific heat of the dispersed phase
- $D$: diameter of the Pitot tube
- $d$: characteristic scale of the system
- $e_T$: isentropic index of the gas phase
- $e_{\text{I},\text{ed}}$: isentropic index of a gas—particle mixture
- $H$: enthalpy
- $K_n$: Knudsen number
- $\lambda_{\text{ed}}$: thermal conductivity of the gas phase
- $m$: mass of an individual particle (or droplet)
- $M_e$: equilibrium Mach number
- $M_f$: frozen Mach number
- $\nu$: dynamic viscosity of the gas phase
- $N$: number of particles (or droplets) per unit volume
- $n$: number of particles (or droplets) per unit mass of the mixture
- $p$: pressure
- $\rho$: total pressure
- $\rho_{\text{eq}}$: equilibrium total pressure
- $\rho_f$: frozen total pressure
- $\rho_{I}$: Prandtl number of the gas phase
- $R$: specific gas constant
- $\rho$: density of the mixture
- $\tau$: relaxation time
- $\phi$: mass fraction of the dispersed phase
- $\phi_s$: density of the gas phase
- $\phi_{\text{eq}}$: density of the dispersed phase

Subscripts

- $\infty$: unperturbed flow variables far upstream of the Pitot tube
- $g$: gas phase
- $d$: dispersed phase (particles or droplets)

2 Conservation Equations

We consider a homogeneous, two-phase mixture consisting of a large number of small, spherical solid particles (or liquid droplets) dispersed in a continuous vapor phase. For simplicity, we assume that the particles (or droplets) are of the same size. Sufficient number density and uniform distribution of the particulate (or droplets) make their interaction with the vapor describable by a continuous variable. We adopt the usual "two-fluid" model and view the particles (or droplets) as providing sources or sinks of mass, momentum, and energy for the vapor, each source term varying continuously in the $x$ direction.

The mass fraction of the dispersed phase, $\phi$, is given by

$$\phi = \phi_s \frac{\rho_{\text{eq}}}{\rho_s}$$

(1)

where there are $n$ particles (or droplets), each of mass $m$, per unit mass of the mixture. The mass of an individual particle is connected to its radius, $r$, and the material density of the dispersed phase, $\rho_s$, via

$$m = \frac{4}{3} \pi r^3 \rho_s$$

(2)

If the vapor density is $\rho_v$, the mixture density $\rho$ (neglecting the volume of the dispersed phase) is

$$\rho = \rho_v (1 - \phi)$$

(3)

and the number of particles (or droplets) $N$ per unit volume is given by

$$N = \frac{\rho_s}{(1 - \phi) \rho_v}$$

(4)

We assume that the vapor phase behaves as a perfect gas with constant isobaric specific heat capacity $c_{p,\text{ed}}$. Thus

$$\rho = \frac{\rho_{\text{eq}} \rho_v}{(1 - \phi)}$$

(5)

and

$$\frac{d\rho}{ds} = c_{p,\text{ed}} \frac{dT}{ds}$$

(6)

where $\rho$ is the specific gas constant and $T_s$ is the temperature of the vapor phase. For the dispersed phase,

$$\frac{d\rho_s}{ds} = c_{p,\text{ed}} \frac{dT_s}{ds}$$

(7)

where $\rho_s$, $T_s$, and $c_{p,\text{ed}}$ are the specific enthalpy, temperature, and the isobaric specific heat of the particles (or droplets).
The basic one-dimensional gasdynamic equations for steady, non-reacting flow for a two-phase system can be written in the usual way (Chen 1965):

**Droplet number conservation:**

\[
\frac{d}{dx}(ANV_d) = 0
\]  
(8)

Continuity:

\[
\frac{d}{dx}(AN_dV_d) + \frac{d}{dx}(ANmV_d) = 0
\]  
(9)

Momentum:

\[
A \frac{d}{dx} \left( \left( h + \frac{V^2}{2} \right) AN_dV_d \right) + \frac{d}{dx} \left( ANmV_d \right) = 0
\]  
(10)

Energy:

\[
\frac{d}{dx} \left[ h_d + \frac{V_d^2}{2} \right] AN_dV_d + \frac{d}{dx} \left( h_m + \frac{V_m^2}{2} \right) ANmV_d = 0
\]  
(11)

where \( h \) is the specific enthalpy, \( V \) is the velocity, \( A \) is the flow area, and \( x \) is the distance along the flow direction. The subscript \( g \) denotes the gas phase and subscript \( d \) refers to the dispersed phase. Equations (1)-(11) are valid for a solid-particle laden gas as well as for a vapor-droplet mixture.

Equations (1)-(11) have to be complemented with rate equations representing interphase transfer processes. For a solid-particle laden gas, we need two equations representing the variation of particle velocity and temperature. These are given by (Guha 1998)

\[
\frac{dV_d}{dx} = \frac{T_d - T_g}{\tau_r}
\]  
(12)

\[
\frac{dT_d}{dx} = \frac{T_d - T_g}{\tau_p}
\]  
(13)

The inertial relaxation time \( \tau_r \) and the thermal relaxation time \( \tau_p \) are given by,

\[
\tau_r = \frac{2\rho_d}{\mu_g} \left[ 1 + 4.5 \left( \frac{Kn}{Le} \right) \right]
\]  
(14)

\[
\tau_p = \frac{\rho_d}{\mu_g} \left[ 1 + 4.5 \left( \frac{Kn}{Le} \right) \right] \Pr
\]  
(15)

where \( \mu_g, \lambda_g, \) and \( Pr \) are the dynamic viscosity, thermal conductivity, and the Prandtl number of the gas phase and \( Kn \) is the particle Knudsen number \( (Kn = L_p/2\nu) \), where \( L_p \) is the molecular mean free path of the gas phase). Equation (13) needs to be replaced by a different equation in the case of vapor-droplet flow (Guha 1998).

In a solid-particle laden gas, there is no interphase mass transfer \( (\text{d}m/\text{d}x = 0) \). The mass conservation equation is, therefore, separately valid for each phase. Equations (1), (4), (8), and (9) can then be combined to give

\[
\frac{\text{d}N_d}{N_p} \frac{V_d}{1 - \gamma_m V_m} = 0
\]  
(16)

where \( \gamma_m \) is the mass fraction of the particles far upstream of the measuring device where the mixture is assumed to be at equilibrium. After some algebraic manipulation of Eqs. (1) -(11), the above equation Eq. (10) can be expressed as:

\[
\frac{\text{d}n_d}{\text{d}x} + \frac{V_d}{\rho_d} \frac{\text{d}V_d}{\text{d}x} + \frac{\text{d}N_d}{\rho_d} \frac{\text{d}V_d}{\text{d}x} = 0
\]  
(17)

Equations (16) and (17) can be combined to give

\[
\frac{\text{d}n_d}{\text{d}x} + \frac{V_d}{\rho_d} \frac{\text{d}V_d}{\text{d}x} + \frac{\text{d}N_d}{\rho_d} \frac{\text{d}V_d}{\text{d}x} = 0
\]  
(18)

Equation (18) is exact. Note that in flows with velocity slip between the phases, the value of \( \gamma_m (1 - \gamma_m) \) does not remain constant along a streamline even in the absence of any interphase mass transfer. Equation (18) shows the correct form of the droplet momentum term (the third term in the LHS of Eq. (18)) which contains the product of the gas phase velocity and \( \text{d}V_d/\text{d}x \). The coefficient \( \gamma_m (1 - \gamma_m) \) remains constant for a specific upstream condition.

### 3 Analytical Solution for Total Pressure Under Non-equilibrium Conditions

For a complete solution of Eqs. (1)-(18), one has to prescribe the variation of the area of the stream tube in front of the Pilot tube (Fig. 1). Since the area variation is not known, the variation of the gas phase velocity is assumed. The vapor velocity is assumed to vary exponentially from the Pilot nozzle,

\[
V_g = V_g(1 - \exp(-x/kD)),
\]  
(19)

where \( V_g \) is the unperturbed velocity far upstream, \( D \) is the pilot diameter, and \( k \) is an empirical constant which may be determined experimentally. Equation (18) shows that as \( x \to 0 \), \( V_d \to V_g \), with \( x \to 0 \). It is found that the nondimensional curve representing the variation in total pressure, as shown later in Fig. 1, does not depend on the value of \( k \). The parameter \( k \) is introduced in the theory to increase its flexibility to match with experiments. If no other data is available then use \( k = 1 \).

For the sake of a simple theory, we linearize Eq. (12) so that an analytical solution can be formulated,

\[
V_d = V_g - \frac{\gamma_m (1 - \gamma_m) V_g}{\tau_p}.
\]  
(20)

**Variation in effective area**

\[
M_{fs} < 1
\]

![Diagram](image)

**Fig. 1. Schematic flow structure in front of a Pilot tube and measured variation in velocity of the gas phase**
Equation (19) is substituted in Eq. (20) and the resulting equation is integrated subject to the boundary condition: at $x = -\infty$, $V_x = V_0$, to give

$$V_x = \frac{1 - \exp(-x/2L)}{1 + S_l},$$  \hspace{1cm} (21)

where the dimensionless Stokes number, $S_l$, is defined by

$$S_l = \frac{\nu V_x}{k D}$$  \hspace{1cm} (22)

The Stokes number, given by (22), is nothing but a Damkohler parameter signifying the importance of the relaxation processes. It can be seen that Eq. (21) prescribes qualitatively correct variation of $V_x$ in different limits. For example in the frozen limit, $S_l = \infty$, $V_x = V_0$ always. In the equilibrium limit, $S_l = 0$, $V_x = V_0$ always. Figure 2 shows the variation of $V_x$ predicted by Eq. (21) for three different values of $S_l$.

Equations (19) and (21) can be substituted in Eq. (18), and the resulting equation can then be integrated between $x = -\infty$ and $x = 0$ in order to find the change in pressure as the gas is brought to rest by the Pitot tube. At a first approximation, we assume that the gas phase is effectively incompressible, so that we can use some average value for its density, $\rho_e$. (Later the effects of such approximation are studied.) In the frozen limit, $dV_x/dx = 0$. Therefore, integrating Eq. (18) between $x = -\infty$ and $x = 0$ in the frozen limit, we obtain

$$\rho_e - \rho_u = 1/2 \rho_e V_e^2,$$  \hspace{1cm} (23)

where $\rho_u$ is the frozen total pressure and $\rho_u$ is the pressure far upstream.

In the equilibrium limit, $V_e = V_0$. Again, integrating Eq. (18) between $x = -\infty$ and $x = 0$ in the equilibrium limit, we obtain

$$\rho_e - \rho_u = 1/2 \rho_e V_e^2,$$  \hspace{1cm} (24)

where $\rho_e$ is the equilibrium total pressure. Now we integrate Eq. (18) between $x = -\infty$ and $x = 0$ in the general nonequilibrium case with $V_e$ and $V_x$ given by Eqs. (19) and (21), respectively.

The result is

$$\rho_e - \rho_u = 1/2 \rho_e V_e^2 \left(1 + \frac{\nu}{1 - \nu} \frac{1}{1 + S_l}\right)$$  \hspace{1cm} (25)

4. Analytical Theory Incorporating Density Changes of the Gas Phase and Thermal Nonequilibrium Effects

In this section, we introduce the effects of density change of the gas phase. For this, we rewrite Eq. (18) as

$$1 \frac{dp}{dx} + \frac{V_x}{RT_e} \frac{dV_x}{dx} = \frac{\rho_u V_x - V_x}{1 - \nu} \frac{dV_x}{dx} = 0.$$  \hspace{1cm} (26)

Equation (26) can be integrated in the same manner as we integrated Eq. (18) in the previous section, if the additional variation in $V_x$ is substituted in Eq. (26). In the frozen limit, the energy equation (11) shows that the temperature of the gas phase, $T_{eq}$, is given by

$$T_{eq} = (\gamma_e T_{eq} + \frac{1}{\gamma_e} V_e^2) / (\gamma_e - 1),$$  \hspace{1cm} (27)

where $T_{eq}$ is the temperature of the gas at $x = 0$. In the equilibrium limit, when the gas and the particles are always at thermal equilibrium, the temperature $T_{eq}$ is again given by the energy equation.

$$T_{eq} = (\gamma_e T_{eq} + \frac{1}{\gamma_e} V_e^2) / (\gamma_e - 1).$$  \hspace{1cm} (28)
where the mixture specific heat, \( c_m \), is given by

\[
\tau = (1 - \gamma) c_v p_m + \gamma c_v p_m.
\]

(31)

Since a complete solution of all the equations to determine \( T_e \) is not feasible in the present context of an analytical theory, it is assumed that, in analogy with Eq. (28), the temperature under nonequilibrium situations varies as

\[
T_e = T_0 + \frac{T_e - T_0}{1 + \alpha}.
\]

(32)

where \( T_0 \) and \( T_e \) are given by Eqs. (29) and (30), respectively. Since \( V_e \) varies with \( \alpha \), Eq. (32) prescribes the variation of \( T_e \) with \( \alpha \). Equations (19), (21), and (32) can be substituted in Eq. (28) and the resulting equation integrated between \( \alpha = 0 \) to \( \alpha = 1 \). The result is

\[
\frac{\partial \rho}{\partial \alpha} = \left( 1 + C \frac{\partial^2}{\partial \alpha^2} \right)^{1/(1+4\alpha)}
\]

(33)

where,

\[
C = 1 + \frac{1}{1 - \gamma} \left[ \frac{1}{1 - \gamma} - 1 \right] \frac{1}{1 - \gamma}.
\]

(34)

and,

\[
\alpha = \frac{1}{1 - \gamma} \left[ \frac{1}{1 - \gamma} - 1 \right] \frac{1}{1 - \gamma}.
\]

(35)

Equation (33) reduces to exact solutions (Ghiba, 1998) in the limits of the frozen and equilibrium flow. In frozen flow, Eq. (33) reduces to

\[
\frac{\rho_m}{\rho} = (1 + 0.5 \gamma = 1.5 \mu_0) \mu_0^{(1-\gamma)}
\]

(36)

where, \( \rho \) is the frozen total pressure, \( \gamma \) is the isentropic exponent of the gas phase alone, and, the far upstream frozen Mach number, \( M_{\infty} \), is calculated from,

\[
M_{\infty} = \frac{V_e}{\theta} = \frac{V_m}{\theta^{1 - \gamma}}
\]

(37)

In the equilibrium limit, Eq. (33) reduces to,

\[
\frac{\rho_m}{\rho} = (1 + 0.5 \gamma = 1.5 \mu_0) \mu_0^{(1-\gamma)}
\]

(38)

where \( \rho_m \) is the equilibrium total pressure, \( \gamma \) is the mixture isentropic exponent, and \( M_{eq} \) is the equilibrium Mach number at the upstream, \( \gamma \) and \( M_{eq} \) are given by

\[
\gamma = \frac{(1 - \gamma) c_v p_m}{\gamma c_v \mu_0 (R - 1) \theta}
\]

(39)

\[
M_{eq} = \frac{V_m}{\gamma^{1/(1 - \gamma)} \theta}
\]

(40)

The nondimensional total pressure \( R_0 \) is \( (p_m - \mu_0)/(p_m - \rho_m) \) can be calculated from Eqs. (33), (35), and (38) and plotted against Stokes number, \( S_t \). The nature of the variation of \( R_0 \) with \( S_t \) is, however, not immediately obvious from the complicated formulae. When the plot of \( R_0 \) versus \( S_t \) obtained in this manner

5 Conclusions

A simple, analytical theory is presented for calculation of the total pressure in two-phase flows. Equation (30) presented here is a very simple relation between the nondimensionalized total pressure, \( R_0 \), and Stokes number, \( S_t \). The equation correctly reduces to the frozen total pressure in the limit of large \( S_t \) (large particle) and to the equilibrium total pressure in the limit of small \( S_t \) (small particles). Under nonequilibrium conditions for intermediate \( S_t \), the prediction of this equation compares well with results from full numerical solution of the gas dynamic equations for multiphase mixtures.

Although Eq. (30) has been derived here by considering subsonic two-phase flow without any interphase mass transfer, it has been shown (Ghiba, 1998) that the same calibration curve of \( R_0 \) versus \( S_t \) could be adopted with acceptable accuracy for diverse multiphase mixtures, vapor-droplet as well as solid particle-laden gases, at a wide range of subsonic and supersonic Mach numbers. Equation (30) can, therefore, be used for inferring flow velocity from measured total pressures in any multiphase mixture at a wide range of flow conditions. The associated problem of interpreting total temperature, equilibrium relations for determining \( p_m \) and \( \rho \), in vapor-droplet mixtures, and, the relation between measured total pressure and entropy production in multiphase flow are discussed at length by Ghiba (1998).

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References


