Thermal Choking Due to Nonequilibrium Condensation

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1 Introduction

It is well known that heat addition causes a reduction in Mach number in supersonic flow and an increase in Mach number in subsonic flow. In other words, heat addition to a flowing fluid drives the Mach number towards unity. Therefore, at a particular flow Mach number, the fluid can absorb a maximum quantity of heat before the Mach number equals unity and the flow becomes thermally choked. It can be easily shown from classical gas dynamics that this critical quantity of heat, \( \dot{Q}_{\text{critical}} \), for simple heat addition (external heat addition without any change in flow cross-sectional area) to an ideal gas is given by (Shapiro, 1953, equation 7.14)

\[
\dot{Q}_{\text{critical}} = \left( \frac{c_p}{c_v} T_0 \right)^2 \left( 2\gamma - 1 \right) \left[ \left( \frac{1 + 2\gamma}{2} \right)^{1.5 - \gamma} - 1 \right]
\]  

(1)

where \( c_p \) is the isobaric specific heat of the ideal gas, \( M_0 \) is the Mach number at which the heat addition begins, \( T_0 \) is the stagnation temperature before heat addition, and \( \gamma \) is the isentropic index (ratio of the two specific heats).

In many practical situations, the energy is supplied internally. For example, when a dry, superheated vapor is expanded rapidly through a convergent-divergent nozzle, liquid droplets do not form as soon as the vapor reaches the saturation temperature. The vapor goes out of thermodynamic equilibrium and attains considerable subcooling, \( T_f \), i.e., the vapor temperature, \( T_f \), falls below the local saturation value, \( T_s \) due to continued expansion. The rate of formation of liquid nuclei is very strongly dependent on the subcooling. Thus when the subcooling becomes appreciable (usually in the divergent part of the nozzle where the flow velocity is supersonic), a very large number of very small nuclei form over a relatively short time. These nuclei grow by exchanging heat and mass with the surrounding subcooled vapor. The resulting release of latent heat is conducted back to the vapor and the vapor temperature quickly rises to the local saturation value (i.e., the subcooling decreases to almost zero). This rapid reversion to equilibrium is generally termed condensation shock and has been the topic of an extremely large number of studies. (The term condensation shock is, in general, a misnomer. Although heat addition in a supersonic flow results in an increase in pressure, the rise is gradual and the Mach number at the end of the condensation zone, in general, remains above unity.)

Similar to the case of external heat addition, the Mach number decreases in the condensation zone (the flow being supersonic). Therefore, for particular combinations of nozzle geometry, supply conditions and the working fluid, the liberation of latent heat could be such that the minimum Mach number becomes unity and the flow is thermally choked. A numerical computation of this limiting case of thermal choking due to nonequilibrium condensation is later shown in Fig. 4. If the inlet total temperature, \( T_{in} \), is reduced any further, keeping the inlet total pressure, \( p_{in} \), fixed, continuous variation of the flow variables is no longer possible and an aerodynamic shock wave appears inside the condensation zone (Burschardt and Filippov, 1970; Guha and Young, 1995).

It is often stated (Wegener and Mack, 1938; Pouring, 1955; Wegener, 1969; Wegener and Ciliento, 1973; Sillings et al., 1987) that supercritical condensation with an initially frozen shock wave occurs when \( \varphi > \varphi_{\text{frozen}} \) where \( \varphi_{\text{frozen}} \) is given by Eq. (1). It has been argued by Guha (1994) that Eq. (1) is not appropriate for a condensing flow primarily for two reasons: (i) In case of a condensation shock, the energy is added
as a result of condensation of a part of the fluid itself. Therefore, the mass flow rate of the condensable vapor changes as the vapor is continuously transferred into the liquid phase. Equation (1), which is derived for external heat addition to an ideal gas, does not take into account this mass depletion. (6) The model, formed through homogeneous nucleation grows at a finite rate by exchanging mass and energy with the surrounding vapor. Therefore, the energy addition due to condensation is not instantaneous and takes place over a short but finite zone. Since condensation shock normally occurs in the diverging section (with dry vapor as inlet), this means the flow area increases between the upstream and downstream of the condensation zone. Equation (1), on the other hand, is derived by assuming heat addition in a constant area duct.

The purpose of the present paper is to derive an expression for the critical amount of heat taking proper account of the above two effects. This would then show the level of incorrects incurred as a result of the approximations in Eq. (1), which is referred extensively in the context of noncondensed condensation. Here we adopt an integral, control volume approach; conditions of choking from differential equations of motion may be found in Oxtoby (1964) and Young (1964). All numerical results presented in this paper are obtained for pure steam, but the analysis is valid for other vapor-droplet mixtures with or without an inert carrier gas.

2 Thermal Choking Considering External Heat Addition With Area Variation

In order to appreciate the effects of area variation and mass depletion on the critical quantity of heat separately, the effects of area variation only are considered in this section. We, therefore, consider external heat addition to an ideal gas in a diverging flow section. We denote the pressure, density, velocity, temperature, sound-speed, flow area and the rate of heat addition per unit mass by \( p, \rho, V, T, a, A, \) and \( q \), respectively. The subscripts 1 and 2 respectively denote the upstream and downstream sections of the control volume within which the heat is added. We can thus write the continuity, momentum and the energy equation across the control volume as,

Continuity

\[ \rho_1 V_1 A_1 = \rho_2 V_2 A_2 = \rho \]

Momentum

\[
\rho_1 (A_1 + \Delta A) + \rho_2 (A_2 - \Delta A) = \rho (V_1 - V_2)
\]

Nomenclature

\( A \) = flow cross-sectional area
\( a \) = speed of sound in gas, or vapor phase
\( A^* \) = area at geometric throat of nozzle
\( c_p \) = isobaric specific heat of gas, or vapor phase
\( \dot{e} \) = specific internal energy of vapor-droplet mixture
\( \gamma \) = isentropic exponent of gas or vapor phase
\( h_s \) = specific enthalpy of vapor-phase
\( h_{lv} \) = specific enthalpy of liquid phase
\( I_{lv} \) = specific enthalpy of liquid phase
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Energy

\[
\Delta h = \frac{1}{\gamma - 1} \frac{M_1^2}{2} + q - \frac{1}{\gamma - 1} \frac{M_2^2}{2}
\]

Equation of state

\[
p = \frac{\gamma R T}{\gamma - 1}
\]

where \( \gamma \) is the specific gas constant. While writing the momentum conservation Eq. (1), the external term \( \Delta p \) has been expressed as \( \text{d}(1/\rho V) \). For the reasons described in Section 4, the adiabatic component of the pressure force on the nozzle wall (upstream) has been approximated by assuming a linear variation of pressure. It is advantageous to employ this approximation in an integral theory. The accuracy of this method has been compared with an exact formulation for an isentropic flow, and later with an accurate time-marching solution of a condensation shock. Both comparisons show that this is a valid approximation for the present problem.

Combining (2) and (3), we obtain

\[
p = \frac{\gamma M_2^2 V}{\gamma + 1}
\]

Substitute (6) in (1) to give

\[
\frac{1}{M_1} \left( \frac{M_1^2}{\gamma - 1} \right) = \frac{1}{M_2} \left( \frac{M_2^2}{\gamma - 1} \right) - \frac{1}{M_2} \left[ \frac{1}{\gamma - 1} \right] \left[ \frac{2 \gamma M_2^2 + 1 - \frac{\Delta A^2}{A^2}}{A^2} \right]
\]

Now we are in a position to assume the accuracy of Eq. (1). Assuming the value of the control volume with subscript 1 to coincide with the geometric throat (with area \( A^* \)) where the Mach number is unity, we can write the variation of local isentropic Mach number as a function of local flow area by letting \( q = 0 \) in (6). The result is

\[
\beta_1 = \frac{1}{\gamma - 1} \frac{M_1^2}{2} + \left[ \frac{1}{\gamma - 1} \right] \left( \frac{2 \gamma M_2^2 + 1 - \frac{\Delta A^2}{A^2}}{A^2} \right)
\]
3 Thermal Choking Considering Condensation With Area Variation and Mass Depletion of the Vapor

In this section, we discuss the case of condensing flow. First, consider a mixture of vapor and liquid phase of the same chemical species. The mass of vapor is continually depleted as a result of condensation. The vapor phase is assumed to obey the perfect gas law. Assuming no velocity slip between the vapor and the liquid phase, the momentum equation may be written from (1) as

\[
(p_v - p_l)A = \dot{m}_{cond}(V_2 - V_1)
\]

where \(\dot{m}_{cond}\) denotes the combined mass flow rate of the vapor and the liquid phase, and \(V_2\) and \(V_1\) are the specific enthalpy of evaporation, \(\gamma\) and \(M\) in Eq. (14) are the frozen speed of sound and the frozen Mach number, respectively.

\[
\frac{1}{\gamma - 1} M^2 = \frac{1}{\gamma - 1} \frac{M^2}{\gamma - 1} + \frac{\gamma p_{in}}{\gamma - 1} A
\]

where \(P_{in}\) is the input pressure and \(\dot{m}_{in}\) is the input mass flow rate.

Substituting (6) and (15) in (13) results in

\[
V_{l} = \frac{\gamma}{A_{1}} \left( \frac{\gamma - 1}{\gamma - 1} M^2 \right) = \frac{V}{\gamma - 1} \frac{A_1}{M_1}
\]

Combining (14) and (16),

\[
\frac{1}{\gamma - 1} M^2 = \frac{1}{\gamma - 1} \frac{M^2}{\gamma - 1} + \frac{\gamma p_{in}}{\gamma - 1} A
\]

Compare Eq. (7) with (16) and (9) with (17) to see the effects of mass depletion. The condition of thermal choking is obtained when the local frozen Mach number is unity (Gula, 1994). Therefore, substituting \(M_1 = 1\) in (7),

\[
(1 + \frac{\gamma - 1}{2}) \left( \frac{V}{\gamma - 1} \frac{A_1}{M_1} \right) = \frac{1}{\gamma - 1} \frac{M^2}{\gamma - 1} + \frac{\gamma p_{in}}{\gamma - 1} A
\]

where \(\frac{\gamma - 1}{2}\) is the critical amount of condensation and \(\bar{A}\) is given by (9). Although the flow is overall adiabatic, we define the critical quantity of heat (for comparisons with previous cases)

\[
\frac{\dot{q}_{cond}}{\bar{A}} = \frac{\gamma - 1}{2} \frac{\gamma M + \frac{\gamma - 1}{2}}{\gamma - 1} M^2
\]

where \(\bar{A}\) is given by (8).

Equation (12) reduces to equation (1) if \(\bar{A}_1 = A_1\). Figure 2 plots the ratio \(\dot{q}_{cond}/\dot{q}_{cond0}\) for different values of the area ratio \(\bar{A}/A_1\). It may be seen that, at low supersonic Mach numbers, even a small increase in area may increase the critical heat many times that predicted by the classical Eq. (6).

\[
\frac{\dot{q}_{cond}}{\dot{q}_{cond0}} = \frac{\gamma - 1}{2} \frac{\gamma M + \frac{\gamma - 1}{2}}{\gamma - 1} M^2
\]

Journal of Fluids Engineering
4 Time-Marching Solution of Condensation Zone

The equations presented in the previous section rest on an integral, control-volume analysis. We assess its accuracy by comparing the prediction of (19) with a full-blown time-marching solution of the governing differential equations for nonequilibrium condensation. Details of the time-marching calculation procedure may be found in Goba and Young (1991). Here we mention the salient points only. Assuming that there is no evaporation slip between the vapor and the liquid droplets (which is usually an acceptable approximation as homogeneous nucleation normally produces very small, submicron size droplets), the gas dynamic equations for inviscid adiabatic incompressible two-phase vapor-droplet flow becomes:

**Continuity**

\[
\frac{d\rho}{dt} + \nabla \cdot (\rho \mathbf{u}) = 0
\]

**Momentum**

\[
\frac{d\rho}{dt} + \nabla \cdot (\rho \mathbf{u}) = 0
\]

**Energy**

\[
\frac{d\rho}{dt} + \nabla \cdot (\rho (\mathbf{u} \cdot \mathbf{u})) = 0
\]

where the vector quantity \( V \) is the common velocity of the two phases, and \( \mathbf{u}, \mathbf{u}_e, \) and \( \mathbf{u}_d \), respectively, the density, the specific enthalpy and the specific internal energy of the mixture.

The mixture density, \( \rho \), is connected to the vapor density, \( \rho_v \), (presuming the volume of the liquid phase and assuming no carrier gas is present) via

\[
\rho = \rho_v(1 - \gamma)
\]

and the mixture specific enthalpy is:

\[
\mathbf{u} = \mathbf{u}_e(1 - \gamma)
\]

where \( \gamma \) and \( \beta \) are the specific enthalpies of the vapor and the liquid phase, respectively.

Equations (20)–(22) are identical to those describing the adiabatic flow of an inviscid single-phase fluid and are valid for unsteady, three-dimensional flows. The differences from an equilibrium calculation are apparent, however, when it is recalled that the wetness fraction \( \gamma \) in Eqs. (23) and (24) is not necessarily the equilibrium value and that \( \beta_0 \) and \( \beta_1 \) in Eq. (24) are evaluated at the respective phase temperatures which are not necessarily equal to the local saturation value \( \beta \). In order to close the set of equations for nonequilibrium condensation, one needs a nucleation rate equation specifying the rate of production of new droplets and a droplet growth law specifying the rate of condensation on existing droplets (providing the nonequilibrium value of \( \gamma \) to be used in (23) and (24)). Both the rates of nucleation and droplet growth depend on the local subcooling, and have been described by Goba and Young (1991).

One of the most effective methods of calculation is to write a computational "black-box" which contains the nucleation and droplet growth equations, and the energy equation in its thermodynamic form. (The equation \( d\beta = d\beta_v = 0 \), derivable from (20)–(22), does not necessarily imply zero entropy increase in multi-phase flow.) Together they furnish the full set of equations that describe completely the formation and growth of liquid droplets in a fluid particle (from a Lagrangian viewpoint) if the pressure-time variation is specified. The pressure-time variation is obtained by time-marching solutions of the conservation equations such as Demet's method (Demet, 1983), extensively used for single-phase calculations in turbomachinery blade rows. In this respect, the thermodynamic aspects of phase-change can be completely divorced from fluid dynamical considerations so that the use of the black-box is
effectively independent of any particular CFD application. Thus established single-phase CFD codes can, rather easily, be modified to deal with non-equilibrium two-phase flow with the above-mentioned modular approach. (The flexibility of this scheme may be appreciated from Guha and Young, 1994 where the same black-box has been grafted into a streamline curvature calculation procedure.)

The development of the computational routines within the black-box represents a considerably major undertaking and has been fully described by Guha and Young (1991). The routines are sufficiently general and robust to deal with any type of nucleating or wet steam flow and (in contrast to many procedures reported in the literature) full details of the droplet size spectrum following nucleation are retained in the calculations. The last aspect is essential for accurate modelling of the nucleation zone. Successive nucleations after the primary one are dealt with as a matter of course should the expansion be sufficiently rapid to generate the high levels of subcooling required. The computational scheme has been validated against measurements of steady (both sub and supercritical) and unsteady condensation shock waves (Guha and Young, 1991).

Here, we present one calculation corresponding to thermal choking.

Figure 4 shows the nozzle employed for calculation. The working fluid is steam and the upstream total pressure, $P_{\infty}$ is 35400 N/m². Keeping the total pressure fixed, the total temperature, $T_{\infty}$ was varied until the condition of thermal choking is obtained. This is found to occur at $T_{\infty} = 256.3$ K. The frozen Mach number profile, $M$, shows that initially steam expands like a dry gas and attains $M = 1$ at the throat of the nozzle. Continued expansion to supercritical velocity ultimately generates sufficient subcooling for appreciable nucleation rate.

The large number of small nuclei grow very fast, and the resulting release of latent heat cools the subcooling and reduces the frozen Mach number. The point of maximum frozen Mach number constitutes the point 1 for the integral analysis described in Sections 2 and 3. Numerical calculations show that the Mach number at point 1, $M_1$, is 1.152.

The released heat is conducted back to the vapor and the frozen Mach number decreases. The inlet stagnation conditions are chosen such that, as the Mach number just reaches unity, the effect of area variation becomes dominant over that of heat addition. The flow, therefore, expands subsequently to supercritical velocity. This is the limiting condition for obtaining a continuous variation in all flow properties. Note that two sonic points exist in the flow field: one at the geometric throat and the second at the point of thermal choking. The point of thermal choking ($M = 1$) constitutes the point 2 in the integral analysis of the previous sections 2 and 3. Numerical calculations show that $M = 1.072$, and the area ratio between Sections 1 and 2 is $A_1/A_2 = 1.028$. We refer the critical amount of heat that transfers to the droplet to the nozzle flow by $\Delta H$. We now may construct Table 1 to compare the results of different integral analyses with the exact solution $\Delta H_{\text{exact}}$.

### Table 1: Comparison of Integral Predictions with Exact Solution

<table>
<thead>
<tr>
<th>Equations</th>
<th>Integral Predictions</th>
<th>Exact Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equation (1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Equation (2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Equation (3)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5 Conclusions

A theory of thermal choking due to nonequilibrium condensation in a nozzle is presented. The theory is based on a single second-order jet. (A differential theory of thermal choking is discussed by Guha and Young, 1994.) It applies to vapor jet flow with or without a carrier gas. The expression for critical heat (or condensation) derived is valid for either supercritical or subcritical flow, and for heat release either in the diverging or in the converging section of the nozzle.

Table 1 shows that the present theory, Eq. (19), is in very good agreement with the full numerical solution of the differential equations of motion (giving the detailed structure of a condensation shock wave leading to thermal choking). The usually quoted Eq. (1) underestimates the critical heat by a factor of three in the example calculation presented. The variation of area across the condensation zone (although small) and the presence of a critical heat in the nozzle flow cannot be neglected in determining the critical heat in condensing nozzle flow.

Acknowledgments

The author is grateful to Cowley & Caisson College, Cambridge, for electing him as a Research Fellow.

References


7th International Symposium on Flow Visualization  
Seattle, Washington USA  
September 11–14, 1995  

Preliminary Announcement and First Call for Papers  
The 7th International Symposium on Flow Visualization will be held September 11–14, 1995 in Seattle, Washington, USA. The Symposium is co-sponsored by the International Flow Visualization Society, the University of Washington, the American Institute of Aeronautics and Astronautics, and The Boeing Company.  

This Symposium is the largest in the series of International Flow Visualization Symposia, first held in Tokyo in 1977, and held every three years since then. The objective of the Symposium is to provide a forum for communication and information exchange in the broad field of Flow Visualization applied over the entire range of disciplines that have come to depend upon this technique. These include experimental and computational fluid dynamics, aerodynamics, chemical and mechanical engineering, bio-medical technology, metallurgy, meteorology, oceanography, food and agricultural technology.  

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  - Laser induced fluorescence  
  - Speckle photography  
  - Shadowgraphy  
  - Schlieren  
  - Interferometry  
  - Particle Image Velocimetry  

- Surface Flow Visualization  
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  - (turb, liquid indicators, liquid crystals)  
  - Thermal and mass transfer interactions  
  - (IR radiometry, sublimation)  
  - Chemical interactions  
  - (reacting indicators, luminous paints)  
  - Graphical Display of Data Sets  
  - Numerical Flow Visualization  

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Authors' Schedule  
Deadline for submission of abstracts: December 10, 1994  
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