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Closed-Form Analytical Solutions for Laminar Natural Convection on Horizontal Plates

A boundary layer based integral analysis has been performed to investigate laminar natural convection heat transfer characteristics for fluids with arbitrary Prandtl number over a semi-infinite horizontal plate subjected either to a variable wall temperature or variable heat flux. The wall temperature is assumed to vary in the form $\overline{T}_w(\overline{x}) - \overline{T}_\infty = a\overline{x}^n$ whereas the heat flux is assumed to vary according to $q_w(\overline{x}) = b\overline{x}^m$. Analytical closed-form solutions for local and average Nusselt number valid for arbitrary values of Prandtl number and nonuniform heating conditions are mathematically derived here. The effects of various values of Prandtl number and the index n or m on the heat transfer coefficients are presented. The results of the integral analysis compare well with that of previously published similarity theory, numerical computations and experiments. A study is presented on how the choice for velocity and temperature profiles affects the results of the integral theory. The theory has been generalized for arbitrary orders of the polynomials representing the velocity and temperature profiles. The subtle role of Prandtl number in determining the relative thicknesses of the velocity and temperature boundary layers for natural convection is elucidated and contrasted with that in forced convection. It is found that, in natural convection, the two boundary layers are of comparable thickness if Pr < 1 or $Pr \approx 1$. It is only when the Prandtl number is large (Pr > 1) that the velocity boundary layer is thicker than the thermal boundary layer. [DOI: 10.1115/1.4024430]

Keywords: natural convection, integral analysis, nonuniform heating, velocity layer thickness, thermal layer thickness

1 Introduction

Natural convection flows are driven by buoyancy force caused by density differences in a fluid. The density difference can be caused by temperature gradients in the fluid. Natural convection may arise in many real situations, such as in the cooling of electronic equipments, solar energy devices, nuclear reactors, heatrecovery systems, room ventilation, crystal growth in liquids, etc. Thus, the phenomenon of natural convection has been studied extensively.

For a vertical semi-infinite plate, analytical study of laminar natural convection for both constant wall temperature and constant heat flux is standard and can be found in Refs. [1–3]. Experimental investigations on both laminar and turbulent natural convection from vertical plates can be found in Refs. [4,5]; these either use constant wall temperature or constant heat flux boundary condition. Correlation equations for Nusselt number variation for natural convection on a vertical plate have been determined analytically [1–3], experimentally [4,5] and on dimensional grounds [6].

Experimental and numerical studies of laminar natural convection from a finite, isothermal horizontal plate have been presented by Fujii and Imura [7], Goldstein and Lau [8] and Clifton and Chapman [9]. Numerical study of laminar free convection heat transfer above an upward facing semi-infinite horizontal heated plate was provided by Yu and Lin [10], Lin et al. [11] and Pretot et al. [12]. Mahajan and Gebhart [13], Pera and Gebhart [14] and Afzal [15] have considered higher order boundary layer effects of natural convection flow over horizontal surfaces. According to Schlichting and Gersten [16], Stewartson [17] was one of the

pioneers to show the existence of this type of boundary layer flow. Stewartson [17] considered an isothermal semi-infinite flat plate and derived the self-similar velocity and temperature profiles for the case of both hot and cold surface facing upward. Later on, Gill et al. [18] interpreted the inconsistency of this solution and showed the existence of similarity solution with either hot surface facing upward or cold surface facing downward. Dayan et al. [19] also showed the nonexistence of self-similar solutions for natural convection underneath a heated horizontal plate and developed an integral analysis for this flow configuration. A similarity analysis has been performed by Rotem and Claassen [20] for free convection over a heated semi-infinite horizontal plate; the reference also includes experimental data and colored visualization pictures. The natural convection boundary layer above a horizontal plate is formed indirectly because of an induced pressure gradient and thus it is termed as "indirect natural convection" [16].

Many engineering heat transfer applications involve cases of laminar natural convection where the surface heating conditions are nonuniform [21]. Solutions for vertical plates with nonuniform surface temperature are given in Refs. [3,21]. For horizontal plates, Chen et al. [22] solved integro-differential equations by finite difference method for two values of the Prandtl number–0.7 and 7. In a recent study [23], similarity theories have been developed for natural convection in fluids with arbitrary Prandtl number on horizontal surfaces for generic power-law variations in wall temperature or wall heat flux.

The aim of the present work is to develop analytical, closedform solutions for boundary layer based models of natural convection flow on horizontal plates. Integral methods constitute a standard solution procedure in boundary layer analysis and are described in textbooks (e.g., Refs. [1–3,16,21]). Integral solutions for natural convection on a vertical plate are easily available in the literature (including the five textbooks mentioned previously). Natural convection on a horizontal plate involves additional

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physical mechanisms not present in the flow on a vertical plate (as explained above). Unlike the boundary layer that forms due to forced convection, the boundary layer on a horizontal plate due to natural convection is such that $\partial p/\partial y \neq 0$ and $\partial p/\partial x$ cannot be neglected inside the boundary layer (even when $\partial p_{\infty}/\partial x$ is zero). Many of such subtle physics of natural convection above a horizontal plate have been included in the theory formulated in this paper. The theory applies to fluids with arbitrary Prandtl number and to nonuniform surface heating conditions. The theory has been generalized for arbitrary orders of the polynomials representing the velocity and temperature profiles. Correlation equations for the local and average Nusselt numbers are also theoretically derived here (Table 1 given later in Section 2.5 summarizes comparative features of the present and previous correlations on Nusselt number variation). In the course of the present formulation the consequences of certain mathematical steps which are common to integral methods for natural convection (including the widely available literature on vertical plates) are examined and the role of Prandtl number in determining the relative extent of the velocity and thermal boundary layers has been established.

2 Mathematical Modeling

2.1 Description of the Physical Problem and Governing Equations. A semi-infinite horizontal flat plate is subjected to a variable wall temperature $\overline{T}_{w}(\bar{x})$ or surface heat flux $q_{w}(\bar{x})$. The heated plate faces upward. The wall temperature or wall heat flux varies as the power of the horizontal coordinate in the form $\overline{T}_{w}(\bar{x}) - \overline{T}_{\infty} = a\overline{x}^{n}$ or $q_{w}(\bar{x}) = b\overline{x}^{m}$. The common mathematical approach for both cases is described below in this section, the part where the mathematical analysis is different for the two boundary conditions is then presented separately in Secs. 2.2 and 2.3.

For mathematical modeling of the physical problem, the flow of fluid is assumed to be incompressible, steady, laminar and two dimensional. The viscous dissipation term in the energy equation is neglected. The Boussinesq approximation for the density variation is applied. The boundary layer equations in dimensional form then become [23] Continuity equation

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{1}$$

 \bar{x} -momentum equation

$$\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}} + \bar{v}\frac{\partial\bar{u}}{\partial\bar{y}} = -\frac{1}{\rho}\frac{\partial\bar{p}}{\partial\bar{x}} + \nu\frac{\partial^2\bar{u}}{\partial\bar{y}^2} \tag{2}$$

 \bar{y} -momentum equation

$$-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial\bar{y}} + \bar{g}\beta(\bar{T} - \bar{T}_{\infty}) = 0$$
(3)

Energy equation

$$\bar{u}\frac{\partial\bar{T}}{\partial\bar{x}} + \bar{v}\frac{\partial\bar{T}}{\partial\bar{y}} = \alpha\frac{\partial^2\bar{T}}{\partial\bar{y}^2} \tag{4}$$

The boundary conditions are

$$\begin{array}{ll} \text{At } \bar{y} = 0, & \bar{u} = 0 \text{ (no slip)}, \ \bar{v} = 0 \text{ (impermeable wall)} \\ & \bar{T}_{w}(\bar{x}) - \bar{T}_{\infty} = a\bar{x}^{n} \quad \text{or} \quad q_{w}(\bar{x}) = b\bar{x}^{m} \\ \text{As } \bar{y} \to \infty, & \bar{u} = 0, \bar{T} = \bar{T}_{\infty}, \bar{p} = \bar{p}_{\infty} \end{array} \right\}$$
(5)

In order to eliminate pressure \bar{p} from Eqs. (2) and (3), Eq. (3) is partially differentiated with respect to \bar{x} , and the resulting equation is then integrated with respect to \bar{y} (noting that $\partial \bar{p}/\partial \bar{x} = 0$ at $\bar{y} \to \infty$). This results in the following combined form of \bar{x} and \bar{y} -

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momentum equations for natural convection over a horizontal plate:

$$\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}} + \bar{v}\frac{\partial\bar{u}}{\partial\bar{y}} = \bar{g}\beta \int_{\bar{y}}^{\infty} \frac{\partial}{\partial\bar{x}}(\bar{T} - \bar{T}_{\infty})d\bar{y} + \nu\frac{\partial^{2}\bar{u}}{\partial\bar{y}^{2}}$$
(6)

The set of Eqs. (1), (6), and (4) needs to be solved, subject to the boundary condition given in Eq. (5).

In an integral boundary layer analysis, the integration with respect to \bar{y} is carried not up to $\bar{y} \to \infty$ but up to the edge of the boundary layer at $\bar{y} = \delta$. All existing integral analyses [1–3,16,21] for natural convection on a vertical plate assume the same thickness (δ) for the velocity and thermal boundary layers. This is to reflect the coupled nature of the velocity and temperature boundary layers in a natural convection problem. This equality is also assumed in the present analysis for horizontal plates; its effects on the results have been discussed later in Sec. 3.1.

To solve the boundary layer equations, the temperature profile is approximated by a parabolic equation of the form

$$\bar{T} = C_1 + C_2 \bar{y} + C_3 \bar{y}^2 \tag{7}$$

The boundary conditions for Eq. (7) are

At
$$\bar{y} = 0$$
, $(\bar{T} - \bar{T}_{\infty}) = a\bar{x}^n$, or, $q_w(\bar{x}) = b\bar{x}^m$
At $\bar{y} = \delta$, $\bar{T} = \bar{T}_{\infty}$, $\frac{\partial \bar{T}}{\partial \bar{y}} = 0$ (8)

The three conditions given in Eq. (8) are used for determining the three coefficients C_1 , C_2 , and C_3 in Eq. (7). C_1 , C_2 , C_3 , and δ are all functions of \bar{x} only. The temperature distribution can therefore be obtained as

$$\frac{\theta}{\theta_{\rm w}} = \frac{\bar{T} - \bar{T}_{\infty}}{\bar{T}_{\rm w} - \bar{T}_{\infty}} = \left(1 - \frac{\bar{y}}{\delta}\right)^2 \tag{9}$$

where,
$$\theta_{\rm w} = (\bar{T}_{\rm w} - \bar{T}_{\infty}) = a\bar{x}^n$$
 for variable wall temperature
 $= \frac{1}{2}b\bar{x}^n\frac{\delta}{k}$ for variable surface heat flux $\begin{cases} 10 \end{cases}$

The profile for velocity \bar{u} may be assumed to be a third-order polynomial given by

$$\bar{u} = C_4 + C_5 \bar{y} + C_6 \bar{y}^2 + C_7 \bar{y}^3 \tag{11}$$

 C_4 , C_5 , C_6 , and C_7 are all functions of \bar{x} only. One needs four boundary conditions to determine the four coefficients in Eq. (11). Three of these conditions can be determined in a straightforward manner

$$\begin{array}{l} \text{at } \bar{y} = 0, \quad \bar{u} = 0 \\ \text{at } \bar{y} = \delta, \quad \bar{u} = 0 \\ \text{at } \bar{y} = \delta, \quad \frac{\partial \bar{u}}{\partial \bar{y}} = 0 \end{array} \right\}$$

$$(12)$$

An additional condition is obtained from Eq. (6)

at
$$\bar{y} = 0$$
, $\nu \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} = -\bar{g}\beta \frac{d}{d\bar{x}} \left[(\bar{T}_{w} - \bar{T}_{\infty}) \frac{\delta}{3} \right]$ (13)

In deriving Eq. (13) (and also a few other equations given below), Leibnitz's rule $(d/dx \int_{k(x)}^{l(x)} f(x, y) dy = \int_{k(x)}^{l(x)} \partial f(x, y) / \partial x dy + f(x, l(x)) dl/dx - f(x, k(x)) dk/dx$) has been used.

Applying the four conditions listed in Eqs. (12) and (13) to the velocity profile given by Eq. (11), one obtains

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$$\frac{\bar{u}}{\bar{u}_{\bar{x}}} = \frac{\bar{y}}{\delta} \left(1 - \frac{\bar{y}}{\delta} \right)^2 \tag{14}$$

where,
$$\bar{u}_{\bar{x}} = \frac{\bar{g}\beta\delta^2}{12\nu}\frac{d}{d\bar{x}}[\theta_{\rm w}\delta]$$
 (15)

Integrating momentum Eq. (6) in the boundary layer within limits $\bar{y} = 0$ and $\bar{y} = \delta$, the momentum integral equation becomes

$$\frac{d}{d\bar{x}} \int_{0}^{\delta} \bar{u}^{2} d\bar{y} = \bar{g}\beta \int_{0}^{\delta} \left\{ \int_{\bar{y}}^{\delta} \frac{\partial}{\partial \bar{x}} (\bar{T} - \bar{T}_{\infty}) d\bar{y} \right\} d\bar{y} - \nu \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)_{\bar{y} = 0}$$
(16)

By following a similar procedure, the integral energy equation can be obtained from Eq. (4) as follows:

$$\frac{d}{d\bar{x}} \int_{0}^{\delta} (\bar{T} - \bar{T}_{\infty}) \bar{u} \, d\bar{y} = -\alpha \left(\frac{\partial \bar{T}}{\partial \bar{y}} \right)_{\bar{y} = 0} \tag{17}$$

Substituting the velocity and temperature profiles from Eqs. (14) and (9) into the integral form of the momentum Eq. (16) and energy Eq. (17) yields

$$\frac{1}{105}\frac{d}{d\bar{x}}(\bar{u}_{\bar{x}}^2\delta) = \bar{g}\beta\frac{d}{d\bar{x}}\left(\frac{\delta^2}{12}\theta_{\rm w}\right) - \frac{\nu\bar{u}_{\bar{x}}}{\delta} \tag{18}$$

$$\frac{1}{30}\frac{d}{d\bar{x}}(\theta_{\rm w}\bar{u}_{\bar{x}}\delta) = 2\frac{\theta_{\rm w}}{\delta}\alpha \tag{19}$$

Equations (18) and (19) are generic equations applicable when either the wall temperature or the wall heat flux is prescribed.

2.2 Power-Law Variation of Wall Temperature. It is assumed that $\bar{u}_{\bar{x}}$ and δ vary according to the following functions:

$$\bar{u}_{\bar{x}} = C_8 \bar{x}^c \quad \text{and} \quad \delta = C_9 \bar{x}^d$$
 (20)

where C_8 , C_9 , c, and d do not depend on \bar{x} .

The expressions for \bar{u}_x and δ from Eq. (20) are substituted in Eqs. (18) and (19), and the first relation of Eq. (10) is used. This results in two simultaneous equations for C_8 and C_9 . It is then argued that for a self-similar solution to exist, both sides of the two equations must be independent of \bar{x} . On equating the exponents of \bar{x} , one obtains c = (2n + 1)/5 and d = (2 - n)/5. The two simultaneous equations can then be solved to give

$$C_9 = \left[\frac{6000}{\bar{g}\beta a(2n+1)(3n+4)} \left\{ \Pr + \frac{4(3n+4)}{21(2n+1)} \right\} \right]^{\frac{1}{5}}$$
$$C_8 = 100\alpha/C_9^2(2n+1)$$

Equation (20) for boundary layer thickness may be written as $\delta/\bar{x} = C_9 \bar{x}^{d-1}$ which, on substitution of the just-determined expressions for *d* and C₉, yields

$$\frac{\delta}{\bar{x}} = \frac{5.6968}{\{(2n+1)(4+3n)\}^{\frac{1}{5}}} \left[\frac{\Pr + \frac{4}{21} \frac{(3n+4)}{(2n+1)}}{\Pr^2 \mathrm{Gr}_{\bar{x}}} \right]^{\frac{1}{5}}$$
(21)

The local heat transfer coefficient from the surface of the plate may be evaluated from $q_w(\bar{x}) = -k(\partial \bar{T}/\partial \bar{y})_{\bar{y}=0} = h_{\bar{x}} (\bar{T}_w - \bar{T}_\infty)$. Evaluating $(\partial \bar{T}/\partial \bar{y})_{\bar{y}=0}$ from Eq. (9) and using the expression for temperature difference $(\bar{T}_w - \bar{T}_\infty)$ from Eq. (10), one obtains

$$\frac{h_{\bar{x}}\bar{x}}{k} = 2\frac{\bar{x}}{\delta}$$
(22)

On substitution of the value of δ/\bar{x} from Eq. (21) in Eq. (22), one obtains the local Nusselt number

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$$Nu_{\bar{x}} = 0.3511\{(2n+1)(4+3n)\}^{\frac{1}{5}} \left[\frac{Pr^{2}Gr_{\bar{x}}}{Pr + \frac{4}{21}\frac{(3n+4)}{(2n+1)}}\right]^{\frac{1}{5}}$$
(23)

The average heat transfer coefficient is given by $\hat{h} = 1/L \int_0^L h d\bar{x}$. Using the expression for local heat transfer coefficient $(h_{\bar{x}})$ from Eq. (22), δ from Eq. (20), and substituting the value of *d* and *C*₉, the average Nusselt number (Nu) over a plate length of *L* can be written as

$$\overset{\wedge}{\mathrm{Nu}} = \frac{1.7555}{(3+n)} \{ (2n+1)(4+3n) \}^{\frac{1}{2}} \left[\frac{\mathrm{Pr}^{2} \mathrm{Gr}_{\mathrm{L}}}{\mathrm{Pr} + \frac{4}{21} \frac{(3n+4)}{(2n+1)}} \right]^{5}$$
(24)

2.3 Power-Law Variation of Wall Heat Flux. For a plate subjected to variable heat flux of the form $q_w(\bar{x}) = b\bar{x}^m$, we assume $\bar{u}_{\bar{x}}$ and δ to vary according to the following functions:

$$\bar{u}_{\bar{x}} = C_{10}\bar{x}^e$$
 and $\delta = C_{11}\bar{x}^f$ (25)

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A similar mathematical procedure as in Sec. 2.2 is now followed, the only difference is that the second relation of Eq. (10) is now used for determining θ_w (the first relation was used in Sec. 2.2). This gives e = m + 1/3, f = 2 - m/6,

$$C_{11} = \left[\frac{2880}{\bar{g}\beta b(m+2)}\frac{k\alpha^2}{(m+1)}\left\{\Pr + \frac{2(m+2)}{7(m+1)}\right\}\right]^{\frac{1}{6}}$$

$$C_{10} = 60\alpha/C_{11}^2(m+1)$$

The expression for boundary layer thickness becomes

$$\frac{\delta}{\bar{x}} = \frac{3.7719}{\{(m+1)(m+2)\}^{\frac{1}{6}}} \left[\frac{\Pr + \frac{2}{7} \frac{(m+2)}{(m+1)}}{\Pr^2 \operatorname{Gr}_{\bar{x}}^*} \right]^{\frac{1}{6}}$$
(26)

The local heat transfer coefficient is evaluated from $h_{\bar{x}} = q_{\rm w}(\bar{x})/(\bar{T}_{\rm w} - \bar{T}_{\infty})$. Substituting the value of $(\bar{T}_{\rm w} - \bar{T}_{\infty})$ from Eq. (10), one obtains

$$\operatorname{Nu}_{\bar{x}} = \frac{h_{\bar{x}}\bar{x}}{k} = 2\frac{\bar{x}}{\delta}$$
(27)

1

With the help of Eq. (26), Eq. (27) becomes

$$Nu_{\bar{x}} = 0.5302\{(m+1)(m+2)\}^{\frac{1}{6}} \left[\frac{\Pr^2 Gr_{\bar{x}}^*}{\Pr + \frac{2}{7} \frac{(m+2)}{(m+1)}} \right]^{\frac{1}{6}}$$
(28)

Following the same mathematical steps given in Sec. 2.2, the average Nusselt number (Nu) over a plate length of *L* is determined to be

$$\overset{\wedge}{\mathrm{Nu}} = \frac{3.1812}{(4+m)} \{ (m+1)(m+2) \}^{\frac{1}{6}} \left[\frac{\mathrm{Pr}^{2} \mathrm{Gr}_{\mathrm{L}}^{*}}{\mathrm{Pr} + \frac{2}{7} \frac{(m+2)}{(m+1)}} \right]^{\frac{6}{2}}$$
(29)

2.4 Summary Results of a Similarity Theory. In a recent work [23], a similarity theory has been developed for steady, laminar natural convection in fluids of arbitrary Prandtl number over a semi-infinite horizontal flat plate for power-law variation in both the wall temperature $(\bar{T}_w(\bar{x}) - \bar{T}_\infty = a\bar{x}^n)$ and the wall heat flux $(q_w(\bar{x}) = b\bar{x}^m)$. The theory automatically captures the thicknesses

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of the velocity and temperature boundary layers in natural convection flow for fluids with arbitrary Prandtl number. The details of the derivation and many physical reflections are given in the cited paper; only the final Nusselt number correlations from Ref. [23] are quoted below so that the predictions of the present integral analysis can be assessed.

When the wall temperature is prescribed as $\overline{T}_{w}(\overline{x}) - \overline{T}_{\infty} = a\overline{x}^{n}$, the similarity theory of Ref. [23] shows that the local Nusselt number (Nu_{\overline{x}}) and the average Nusselt number (Nu) are, respectively, given by

$$Nu_{\bar{x}} = -g' \ (0) (Gr_{\bar{x}})^{\frac{1}{5}}$$
(30)

$$\hat{N}u = -\frac{5}{n+3} (Gr_L)^{\frac{1}{5}} g'(0)$$
(31)

g'(0) in Eqs. (30) and (31) specifies the surface heat flux and is defined as $g' = dg/d\eta_{wt}$, where η_{wt} is the similarity variable given by $\eta_{wt} = \bar{y}(\text{Gr}_{\bar{x}})^{1/5}/\bar{x}$. g'(0) depends on the Prandtl number Pr and the exponent *n*. The function *g* specifies the self-similar temperature profile in the similarity theory and is determined (together with two other functions giving velocity and pressure) by numerically solving a system of three nonlinear coupled ordinary differential equations with variable coefficients.

When the wall heat flux is prescribed as $q_w(\bar{x}) = b\bar{x}^m$, the similarity theory of Ref. [23] shows that the local Nusselt number $(Nu_{\bar{x}})$ and the average Nusselt number (Nu) are, respectively, given by

$$Nu_{\bar{x}} = \frac{1}{(18)^{\frac{1}{6}}G(0)} (Gr_{\bar{x}}^*)^{1/6}$$
(32)

$$\hat{N}u = \frac{6}{m+4} \frac{1}{(18)^{\frac{1}{6}} G(0)} \left(\text{Gr}_{\text{L}}^* \right)^{1/6}$$
(33)

G(0) in Eqs. (32) and (33) represents the wall temperature for the case of specified wall heat flux; it depends on the Prandtl number Pr and the exponent *m*. The function *G* specifies the self-similar temperature profile in the similarity theory and is determined (together with two other functions giving velocity and pressure) by numerically solving a system of three nonlinear coupled ordinary differential equations with variable coefficients. Details are available in Ref. [23].

2.5 Comparison of the Present Integral Analysis and the Similarity Theory of Ref. [23]. For the case of specified wall temperature, the present integral analysis gives Eq. (23) for the variation of Nusselt number and the similarity theory of Ref. [23] gives Eq. (30). A comparison of Eqs. (23) and (30) show that they can be combined through a common functional form

$$Nu_{\bar{x}} = f_1(n, Pr)(Gr_{\bar{x}})^{\frac{1}{5}}$$
 (34)

Similarly, for the case of specified wall heat flux, a comparison of Eqs. (28) and (32) show that they can be combined through a common functional form

$$Nu_{\bar{x}} = g_1(m, Pr)(Gr^*_{\bar{x}})^{\frac{1}{6}}$$
 (35)

The functions $f_1(n, Pr)$ and $g_1(m, Pr)$ in Eqs. (34) and (35) are summarized in Table 1 for ready reference.

It is interesting to consider Eqs. (23), (24), (30), and (31) to note that the integral analysis and the similarity theory give the same ratio for $Nu/Nu_{\tilde{x}}$

$$\overset{\wedge}{\mathrm{Nu}}/\mathrm{Nu}_{\bar{x}} = \frac{5}{n+3} \left(\frac{\mathrm{Gr}_{\mathrm{L}}}{\mathrm{Gr}_{\bar{x}}}\right)^{\frac{1}{5}}$$
(36)

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Table 2 Values of $f_1(n, Pr)$ in Eq. (34) for an isothermal plate for various values of Pr

Pr	Integral analysis (present study) $\chi = 2, \lambda = 3$	Numerical solution by Yu and Lin [10]	Numerical solution by Pera and Gebhart [14]	Integro-differential analysis by Chen et al. [22]	Similarity analysis by Samanta and Guha [23]
0.01	0.0773	0.0708	n/a	n/a	0.0876
0.1	0.1900	0.1748	n/a	n/a	0.1961
0.7	0.3723	0.3491	n/a	0.3554	0.3543
1	0.4137	0.3897	0.3940	n/a	0.3895
10	0.7235	0.6962	n/a	n/a	0.6770
100	1.1619	1.1224	n/a	n/a	1.0896

Table 3 Values of $f_1(n, Pr)$ in Eq. (34) for various values of n when Pr = 1

n	Integral analysis (present study) $\chi = 2, \lambda = 3$	Similarity analysis [23]
0	0.4137	0.3895
1	0.5997	0.6466
3	0.8146	0.9327
5	0.9655	1.1220

Similarly, for the prescribed wall heat flux, it is noted through a consideration of Eqs. (28), (29), (32), and (33) that the integral analysis and the similarity theory give the same ratio for $Nu / Nu_{\bar{r}}$

$$\overset{\wedge}{\mathrm{Nu}}/\mathrm{Nu}_{\bar{x}} = \frac{6}{m+4} \left(\frac{\mathrm{Gr}_{\mathrm{L}}^*}{\mathrm{Gr}_{\bar{x}}^*}\right)^{\frac{1}{6}}$$
(37)

It is to be noted that the correct functional dependence of the Nusselt number on Grashof number, i.e., the index 1/5 for wall temperature case and the index 1/6 for the wall heat flux case, that comes out of the present mathematical analysis is also borne out by the similarity theory [23] and the numerical solution of integro-differential equations [22].

2.6 Special Features of the Present Solution. The distinctive features of the present analytical solutions in the context of available literature can be appreciated from a study of Table 1. The heat transfer correlation equations (Eqs. (23), (24), (28), and (29)) have five important characteristics all of which are not simultaneously present in the previous work on natural convection over a horizontal plate: (i) the correlation equations are mathematically derived from the equations for conservation of mass, momentum and energy, (ii) they are explicit analytical relations, (iii) they produce the correct functional dependence of the Nusselt number on Grashof number (the index 1/5 for wall temperature case and the index 1/6 for the wall heat flux case), (iv) they can be applied with reasonable accuracy over a wide range of Prandtl number, (v) they are valid for nonuniform wall temperature or wall heat flux. The features of previously available correlations [10,11,14,22] are summarized in Table 1. All of these correlations were found by curve-fitting through numerical solutions, they

Table 5 Values of $g_1(m, Pr)$ in Eq. (35) for various values of m when Pr = 1

m	Integral analysis (present study) $\chi = 2, \lambda = 3$	Similarity analysis [23]	
0	0.5519	0.5626	
1	0.6735	0.7285	
3	0.8302	0.9301	
5	0.9422	1.0680	

apply either to constant surface temperature or to constant heat flux case, some apply only over a restricted range of Prandtl number. The similarity solutions of Ref. [23] are valid for arbitrary Prandtl number and nonuniform surface heating conditions but explicit correlations showing algebraic dependence of the Nusselt number on Prandtl number are not available from the similarity theory whereas such algebraic dependence is available in the present work.

3 Results and Discussion

Table 2 presents a comparative study of the characteristic numerical values of the function $f_1(n, \text{Pr})$ appearing in Eq. (34) for an isothermal plate (i.e., n = 0) for various values of Prandtl number. Table 3 shows the numerical values of $f_1(n, \text{Pr})$ for various values of n when Pr = 1. Table 4 presents a comparative study of the characteristic numerical values of the function $g_1(m, \text{Pr})$ appearing in Eq. (35) for a constant-heat-flux plate (i.e., m = 0) for various values of $g_1(m, \text{Pr})$ for various values of m when Pr = 1. From Tables 2 and 4, it is seen that, as the Prandtl number increases, the value of $f_1(n, \text{Pr})$ or $g_1(m, \text{Pr})$ and hence that of $\text{Nu}_{\bar{x}}$ increases. From Tables 3 and 5, it is seen that $\text{Nu}_{\bar{x}}$ increases with increasing value of n or m. The difference between the results of the integral analysis and the similarity analysis increases as the value of n or m increases.

A careful observation of Tables 2 and 4 reveals that the ratio Nu_{UHF}/Nu_{UWT} is always greater than 1, where the subscripts UHF and UWT, respectively, denote "uniform heat flux" and "uniform wall temperature." The ratio decreases with increase in Prandtl number. These behaviors of the ratio Nu_{UHF}/Nu_{UWT} , and the role played by the values of the exponents *n* or *m*, may be understood quantitatively from a study of the explicit, closed-form correlations that have been derived in this paper, viz., Eqs. (23) and (28).

Table 4 Values of $g_1(m, Pr)$ in Eq. (35) for a constant heat flux plate for various values of Pr

Pr	Integral analysis (present study) $\chi = 2, \lambda = 3$	Numerical solution by Lin et al. [11]	Integro-differential analysis by Chen et al. [22]	Similarity analysis by Samanta and Guha [23]
0.01	0.1403	0.1326	n/a	0.1689
0.1	0.2952	0.2816	n/a	0.3276
0.7	0.5077	0.5011	0.5203	0.5216
1	0.5519	0.5492	n/a	0.5626
10	0.8655	0.8907	n/a	0.8788
100	1.2810	1.3261	n/a	1.3200

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Fig. 1 Local Nusselt number versus local Grashof number for an isothermal horizontal plate for Pr = 0.7: assessment of the present integral analysis (— Present integral analysis ($\lambda = 3, \chi$ = 2), + Numerical solution [10], \diamond Numerical solution [14], \blacktriangle Integro-Differential analysis [22], • Similarity solution [23], × Experimental correlation [24], \Box Experimental correlation [25])



Fig. 2 Local Nusselt number versus modified local Grashof number for a horizontal plate with constant heat flux for Pr = 0.7: assessment of the present integral analysis (—— Present integral analysis ($\lambda = 3$, $\chi = 2$), + Numerical solution [11], \triangle Integro-Differential analysis [22], • Similarity solution [23])

Figure 1 depicts the variation in local Nusselt number $(Nu_{\bar{x}})$ with local Grashof number $(Gr_{\bar{x}})$ for an isothermal plate when Pr = 0.7. The prediction of the present integral method is compared with other theoretical and experimental results available in the literature [10,14,22-25]. The integral analysis agrees well with more complex theoretical models. The experimental results show higher values for the Nusselt number and, Fishenden and Saunders [24] had proposed 1/4th power-law variation with Grashof number (instead of the 1/5th power predicted by all theories including Eq. (34)); this aspect has been noted by Goldstein and Lau [8]. The equivalent $Nu_{\bar{x}}$ versus $Gr_{\bar{x}}$ relation that may be constructed from the experiments of Ref. [8] based on naphthalene sublimation would follow the same trend of experimental results shown in Fig. 1 in the approximate range of Grashof number $10^2 < Gr_{\bar{x}} < 10^4$. The experiments, however, confirm the existence of this type of boundary layer flow above a heated plate. It



Fig. 3 Nondimensional velocity and temperature profiles for an isothermal horizontal plate for Pr = 0.01 (---- similarity solution [23]; ---- third order velocity profile λ = 3, second order temperature profile χ = 2; ---- eighth order velocity profile λ = 8, second order temperature profile χ = 2)





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is possible to carry out stability analysis of the laminar flow solution and find out the magnitudes of critical Grashof number; such aspects are not within the scope of this study. While this paper was in press, we traced an early attempt of applying integral analysis to isothermal horizontal plates [26], but the solution underpredicts values of the Nusselt number (as compared to the present isothermal results) by a factor $3^{1/5}$. Details of the derivation are not given in Ref. [26], but one possibility is that the \bar{x} -derivative of the temperature term was inappropriately taken outside the integral sign while evaluating Eq. (16), which would result in the same error.

Figure 2 presents the variation of local Nusselt number $(Nu_{\bar{x}})$ with modified local Grashof number $(Gr_{\bar{x}})$ for a constant heat flux plate when Pr = 0.7. A comparison with the existing correlations available in the literature [11,22,23] shows that the present integral analysis gives good prediction of $Nu_{\bar{x}}$.

3.1 Comments on the Relative Thicknesses of the Velocity and Thermal Boundary Layers. Three major approximations in an integral analysis are: the assumed velocity profile, temperature profile, and the relative thicknesses of the two boundary layers. The last aspect is discussed here briefly while the choice of profiles has been analyzed in Sec. 3.2. All published integral analyses of natural convection flow on a vertical surface [e.g., Refs. 1-3,16,21] assume that the thicknesses of the two boundary layers are equal ($\delta_{\rm V} = \delta_{\rm T} = \delta$). This assumption has also been used in developing the present integral analysis for horizontal plates. The advantage of this assumption is that closed-form analytical solutions with reasonable accuracy can be formulated; however, one disadvantage is that, with one less variable, one of the equations has to be discarded (i.e., not used in the final solutions): usually the original equation defining $\bar{u}_{\bar{x}}$ is discarded. In the present formulation, Eq. (15) is not used in the subsequent analysis, and $\bar{u}_{\bar{x}}$ is calculated by Eq. (20) giving $\bar{u}_{\bar{x}} = C_9 \bar{x}^c$: it is found that, in replacing Eq. (15) by Eq. (20), the value of the exponent c remains unaltered but the value of the coefficient C_9 gets changed.

The role of Prandtl number is also subtle. In the case of forced convection heat transfer [1,16,27], it is found that $\delta_V / \delta_T \sim Pr^{1/2}$ for fluids with low Prandtl number and $\delta_V / \delta_T \sim Pr^{1/3}$ for fluids with medium or high Prandtl number. Therefore, for forced convection, $\delta_{\rm V} < \delta_{\rm T}$ when $\Pr < 1$, $\delta_{\rm V} \sim \delta_{\rm T}$ when $\Pr \sim 1$, $\delta_{\rm V} > \delta_{\rm T}$ when Pr > 1. This behavior is consistent with the definition of Prandtl number which is defined as the ratio of the momentum and thermal diffusivities. It may be construed that this role of Prandtl number remains the same for the case of natural convection also. A comment made by one of the reviewers, which is similar to the statement made in Ref. [1, p. 525] (" $\delta_V \approx \delta_T$ only if $Pr \approx 1$ "), has drawn our attention to this possible implication. Ghiaasiaan [28, p. 278) has also commented that the relative thickness of the two boundary layers follows the same trend in forced and natural convection. An opposite qualitative argument could also be formed that, since the two boundary layers are coupled in natural convection, they would be of similar thickness at all Prandtl numbers (as may be implied in Fig. 10.16 in Schlichting and Gersten [16, p. 281]).

In order to settle this issue qualitatively and quantitatively, detailed calculations have been performed with the help of the recently developed similarity theory for natural convection on horizontal plates [23]. For the sake of brevity we only report the main results in Figs. 3–5. The similarity variable is plotted in the

abscissa; it is shown in [23] that, for an isothermal horizontal plate, the similarity variable is given by $\eta_{wt} = \bar{y} (\text{Gr}_{\bar{x}})^{1/5} / \bar{x}$. It is found that, in natural convection, the two boundary layers are of comparable thickness if $\text{Pr} \le 1$ or $\text{Pr} \approx 1$. It is only when the Prandtl number is large (Pr > 1) that the velocity boundary layer is thicker than the thermal boundary layer. In natural convection, the velocity boundary layer is never less thick than the thermal boundary layer can, however, become thicker than the thermal boundary layer can, however, become thicker than the thermal boundary layer, when the Prandtl number is very large, because natural convection velocity may persist away from the wall due to shear force and inertia (even when buoyancy is absent).

From the above discussion it can be concluded that the assumption $\delta_V / \delta_T \approx 1$ made in the integral analysis of natural convection is reasonable, inaccuracy in the results due to this assumption is expected only at high Prandtl numbers. Comparison with the predictions of the similarity theory or other data given in the tables and figures of this paper, however, shows that the percentage error of the Nusselt number predicted by the integral theory is lower at high Prandtl number than that at low Prandtl number.

3.2 Effects of Various Choices for Velocity and Temperature Profiles. In this paper, the algebraic details of the mathematical theory are given only for the case of cubic velocity profile (Eq. (14)) and a quadratic temperature profile (Eq. (9)). Other choices for the profiles are also possible. In fact, the boundary condition $\partial \bar{p}/\partial \bar{x} = 0$ at $\bar{y} \to \infty$ was used in deriving the combined momentum Eq. (6). This boundary condition implies, according to Eq. (2), that $\partial^2 \bar{u}/\partial \bar{y}^2 \to 0$ at $\bar{y} \to \infty$. Similarly, Eq. (4) implies that $\partial^2 \bar{T}/\partial \bar{y}^2 \to 0$ at $\bar{y} \to \infty$ (since $\bar{u} \to 0$ and $\partial \bar{T}/\partial \bar{y} \to 0$). Thus, it seems that a cubic profile for the temperature and a fourth-order polynomial for the velocity should be the natural choice for an integral analysis of free convection on horizontal plates. Even higher order polynomials can be used by setting higher order derivatives of velocity ($\partial^{l} \bar{u}/\partial \bar{y}^{l}$) and temperature ($\partial^{l} \bar{T}/\partial \bar{y}^{l}$) to zero at $\bar{y} \to \infty$, where l = 3, 4, ..., etc.

By systematically applying the various boundary conditions at $\bar{y} = 0$ and $\bar{y} \to \infty$, it is found that the various velocity and temperature profiles can be generically written as

$$\frac{\bar{u}}{\bar{u}_{\bar{x}}} = \frac{\bar{y}}{\delta} \left(1 - \frac{\bar{y}}{\delta} \right)^{\lambda - 1}$$
(38)

$$\frac{\theta}{\theta_{\rm w}} = \frac{\bar{T} - \bar{T}_{\infty}}{\bar{T}_{\rm w} - \bar{T}_{\infty}} = \left(1 - \frac{\bar{y}}{\delta}\right)^{\chi} \tag{39}$$

where λ and χ are integers, and denote the order of the polynomial used, respectively, for the velocity and temperature profile.

By repeating the integral analysis for various velocity and temperature profiles, for prescribed wall temperature $\overline{T}_{\rm w}(\bar{x}) - \overline{T}_{\infty} = a\overline{x}^n$, we found that the Nusselt number can be expressed by the generic expression

$$\operatorname{Nu}_{\bar{x}} = a_1 \{ (2n+1)(4+3n) \}^{\frac{1}{5}} \left[\frac{\operatorname{Pr}^2 \operatorname{Gr}_{\bar{x}}}{\operatorname{Pr} + a_2 \frac{(3n+4)}{(2n+1)}} \right]^{\frac{1}{5}}$$
(40)

Table 6 Values of f_1 (*n*, Pr) for various velocity profiles for isothermal plate (n = 0 in Eq. (40)) for quadratic temperature profile ($\chi = 2$ in Eq. (39))

Pr	Similarity solution [23]	Third order $\lambda = 3, a_1 = 0.3511, a_2 = 4/21$	Fourth order $\lambda = 4, a_1 = 0.3282, a_2 = 1/9$	Sixth order $\lambda = 6, a_1 = 0.2946, a_2 = 8/143$	Eighth order $\lambda = 8, a_1 = 0.2707, a_2 = 11/306$
0.01	0.0876	0.0773	0.0803	0.0824	0.0823
0.7	0.3543	0.3723	0.3655	0.3425	0.3204
100	1.0896	1.1619	1.0869	0.9762	0.8970

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Table 7 Values of $f_1(n, Pr)$ for various velocity profiles for isothermal plate (n = 0 in Eq. (40)) for cubic temperature profile ($\chi = 3$ in Eq. (39))

Pr	Similarity solution [23]	Third order $\lambda = 3, a_1 = 0.4099, a_2 = 2/5$	Fourth order $\lambda = 4, a_1 = 0.3869, a_2 = 2/9$	Sixth order $\lambda = 6, a_1 = 0.3519, a_2 = 15/143$	Eighth order $\lambda = 8, a_1 = 0.3259, a_2 = 11/170$
0.01	0.0876	0.0779	0.0826	0.0871	0.0886
0.7	0.3543 1.0896	0.3969 1.3542	0.4034 1.2800	0.3936 1.1655	0.3759 1.0796



Fig. 6 Variation in nondimensional heat transfer coefficient with Prandtl number: assessment of the present integral analysis (-Present integral analysis ($\lambda = 3$, $\chi = 2$), \triangle Numerical solution (isothermal plate) [10], ▲ Numerical solution (constant heat flux plate) [11], • Similarity solution (isothermal plate) [23], ○ Similarity solution (constant heat flux plate) [23])

The values of a_1 and a_2 in Eq. (40) for various profiles are given in Tables 6 and 7.

For prescribed surface heat flux $q_w(\bar{x}) = b\bar{x}^m$, we found that the Nusselt number can be expressed by the generic expression

$$\mathrm{Nu}_{\bar{x}} = a_3 \{ (m+1)(m+2) \}^{\frac{1}{6}} \left[\frac{\mathrm{Pr}^2 \mathrm{Gr}_{\bar{x}}^*}{\mathrm{Pr} + a_4 \frac{(m+2)}{(m+1)}} \right]^{\frac{1}{6}}$$
(41)

For quadratic temperature profile ($\gamma = 2$), the values of a_3 and a_4 in Eq. (41) for third, fourth, sixth, and eighth order velocity profiles are, respectively, given by $(a_3 = 0.5302, a_4 = 2/7)$, $(a_3 = 0.5013, a_4 = 1/6),$ $(a_3 = 0.4582, a_4 = 12/143),$ and $(a_3 = 0.4269, a_4 = 11/204).$

A study of Figs. 3-5 and Tables 6 and 7 shows that, with polynomials of higher order λ , the velocity profile close to the wall becomes steeper and the velocity approaches more gradually to zero at $\bar{y} = \delta$. At low values of Prandtl number (Pr < 1), higher order velocity profiles thus improve the prediction of Nusselt number (and skin friction coefficient). At $Pr \approx 1$, and at Pr > 1, very high orders of the velocity profile may, however, make the velocity gradient at wall greater than the correct value; this results in underprediction in Nusselt number (and overprediction in skin friction coefficient). For the prediction of Nusselt number for isothermal horizontal plate, the best compromise for all Prandtl numbers is to use the second-order temperature profile ($\chi = 2$) and fourth-order velocity profile ($\lambda = 4$).

Figure 6 depicts the variations in local Nusselt number (Nu_{\bar{x}}) with Prandtl number (Pr) both for an isothermal and an iso-heatflux plate. The composite variables $Nu_{\bar{x}}(Gr_{\bar{x}})^{-1/5}$ and $Nu_{\bar{x}}(Gr_{\bar{x}}^*)^{-1/6}$ are plotted as the ordinates so that data generated by comprehensive computations can be presented in a concise

manner. It is found that present closed-form Nusselt number correlations agree well with results of similarity theory [23] and other previous work at all Prandtl numbers (for both constant temperature and constant heat flux cases).

4 Conclusion

In this paper, a boundary layer based integral analysis of steady, laminar natural convection over a semi-infinite horizontal flat plate for power-law variation in both the wall temperature $(\bar{T}_{w}(\bar{x}) - \bar{T}_{\infty} = a\bar{x}^{n})$ and the surface heat flux $(q_{w}(\bar{x}) = b\bar{x}^{m})$ has been made. The present theory results in explicit, closed-form algebraic solutions for the boundary layer thickness δ/\bar{x} and the local Nusselt number Nu_x: Eqs. (21) and (40) for prescribed wall temperature show that $\delta/\bar{x} \sim 1/(Gr_{\bar{x}})^{1/5}$, Nu_{$\bar{x}} <math>\sim (Gr_{\bar{x}})^{1/5}$, and Eqs. (26) and (41) for prescribed wall heat flux show that $\delta/\bar{x} \sim 1/(Gr_{\bar{x}}^*)^{1/6}$, Nu_{$\bar{x}} <math>\sim (Gr_{\bar{x}}^*)^{1/6}$. The value of Nu /Nu_{\bar{x}} is speci-</sub></sub> fied by Eqs. (36) and (37). The present theory shows that $Nu_{\bar{x}}$ increases with increasing values of exponent n or m, and with increasing Prandtl number. The explicit relations derived here are valid for a wide range of values of Pr and n or m, and compare well with the results of previous works [10,11,14,22,23]. Table 1 summarizes comparative features of present as well as previous works. The particular cases of constant wall temperature and constant heat flux can be obtained by, respectively, substituting n = 0or m = 0 in the present solutions. It can be shown that the constant heat flux case results in a variation of surface temperature such that n = 1/3. Similarly in order to maintain constant surface temperature, the surface heat flux must vary such that m = -2/5.

It is shown that, in natural convection, the velocity and thermal boundary layers are of comparable thickness if $Pr \le 1$ or $Pr \approx 1$. It is only when the Prandtl number is large (Pr > 1) that the velocity boundary layer is thicker than the thermal boundary layer. Thus, the role of Prandtl number in natural convection is quite different from that in forced convection.

For the prediction of Nusselt number for isothermal horizontal plate (n=0), the best compromise for all Prandtl numbers is to use the second-order temperature profile ($\chi = 2$) and fourth-order velocity profile ($\lambda = 4$), i.e., $a_1 = 0.3282$, $a_2 = 1/9$ should be used in Eq. (40). For constant heat flux case (m=0), the second-order temperature profile ($\chi = 2$) and third-order velocity profile ($\lambda = 3$) give good agreement at all Prandtl numbers, i.e., $a_3 = 0.5302$, $a_4 = 2/7$ should be used in Eq. (41).

Nomenclature

- a = dimensional constant in the power-law variation of wall temperature
- b = dimensional constant in the power-law variation of wall heat flux
- $f_1(n, \Pr) = a$ function defined in Eq. (34)
- $\bar{g} = \text{gravitational acceleration}$
- $g_1(m, \Pr) = a$ function defined in Eq. (35)
 - $Gr_{\bar{x}} = Grashof number defined as \bar{g}\beta [\bar{T}_w(\bar{x}) \bar{T}_\infty]\bar{x}^3/\nu^2$
 - $Gr_{\bar{x}}^* = modified Grashof number defined as \bar{g}\beta q_w(\bar{x})\bar{x}^4/k\nu^2$
 - $h_{\bar{x}} = \text{local heat transfer coefficient, } q_{w}(\bar{x})/[\bar{T}_{w}(\bar{x}) \bar{T}_{\infty}]$ \hat{h} = average heat transfer coefficient given by $\hat{h} = 1/L \int_0^L h d\bar{x}$
 - k = thermal conductivity of the fluid
 - L = reference length of the plate in \bar{x} direction

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- m = exponent in the power-law variation of wall heat flux
- n = exponent in the power-law variation of wall
- temperature
- $N\mu_{\bar{x}} = \text{local Nusselt number, } h\bar{x}/k$
- $\hat{N}u = average Nusselt number, \hat{h}L/k$
- $\bar{p}_{\infty} =$ static pressure in the undisturbed fluid
- $\Pr = \Pr$ and the number, ν/α
- $\bar{T} =$ fluid temperature
- \bar{T}_{∞} = temperature of the quiescent fluid
- \bar{u} = velocity component in the \bar{x} direction
- $\bar{u}_{\bar{x}}$ = velocity scale derived in Eq. (15)
- $\bar{u}_{0,s}$ = velocity scale obtained from similarity theory [23], $\bar{u}_{0,s} = \nu G r_z^{2/5} / \bar{x}$
 - $\bar{v} = normal velocity component$
- $\bar{x} =$ horizontal coordinate
- $\bar{y} =$ vertical coordinate

Greek Symbols

- α = thermal diffusivity
- β = coefficient of thermal expansion at the reference temperature
- $\chi =$ order of polynomial for temperature (Eq. (39))
- δ = boundary layer thickness derived in Eqs. (21) and (26), respectively
- η_{wt} = similarity variable defined as $\eta_{wt} = \bar{y}(Gr_{\bar{x}})^{1/5}/\bar{x}$
- λ = order of polynomial for velocity (Eq. (38))
- $\nu =$ kinematic viscosity
- $\rho = \text{density of fluid}$
- θ = temperature difference given by $(\bar{T} \bar{T}_{\infty})$
- $\theta_{\rm w}$ = temperature difference defined in Eq. (10)

Subscripts

- w = condition at the wall
- $\infty =$ condition in undisturbed fluid

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